Résonances espace-temps et instabilités en optique non-linéaire

Space-time resonances generate high-frequency instabilities in the two-fluid Euler-Maxwell system

Éric Dumas

Grenoble

with Lu Yong (Nanjing) and Benjamin Texier (Lyon)

Nancy, Journées EDP IECL, 29 March 2022

Physical context: inertial confinement fusion

from wikipedia:

Inertial confinement fusion is a type of *fusion energy research* that attempts to initiate nuclear fusion reactions by heating and compressing a fuel target, typically in the form of a pellet that most often contains a mixture of deuterium and tritium. Typical fuel pellets are about the size of a pinhead. To compress and heat the fuel, energy is delivered to the outer layer of the target using high-energy beams of laser light.

Photo: LLNL, 1984, source: wikipedia

Physical context: inertial confinement fusion

from wikipedia:

Throughout the 1980s and '90s, many experiments were conducted in order to understand the complex interaction of high-intensity laser light and plasma. These led to the design of newer machines, much larger, that would finally reach ignition energies.

- LLNL, USA
- LMJ, CEA, Bordeaux

  from the CEA webpage: "très grand instrument ... mis en service en 2014"
- Institute for Laser engineering, Osaka

Physical context: inertial confinement fusion

- Huge equipments, but...
- experiments are not satisfactory:
  "Laser-plasma instabilities inhibit the deposition of energy ... [New broad-bandwith lasers could potentially] suppress high-frequency instabilities like [...] stimulated Râman scattering"
  White paper on opportunities in plasma physics, 2019.
- Râman effect: predicted by Smekal in 1923, observed by Râman and Krishnan (and Mandelstam, independently) in 1928.

Plan:

- *the framework: Euler-Maxwell and 3 scales geometric optics*

  starting from spectral properties: the linearized equations around 0

- validation of the instability for the nonlinear system
- three viewpoints on *resonances and space-time resonances*
- the symbolic flow approach to the spectral instability problem

General mathematical context:

- quasi-linear or semi-linear hyperbolic systems depending on small (large) parameters
- highly-oscillating data (*"geometric optics"*)
- References: Lax, Keller, Hunter, Majda, Rosales, Newell, Moloney, Joly, Métivier, Rauch, Guès, Colin, Lannes, Williams, Coulombel...

Specific mathematical context:

- *Geometric optics.*

  Highly-oscillatory solutions to hyperbolic systems.

  Supercritical geometric optics: [Joly-Métivier-Rauch, 2000]

  $\sim$ long time

  compatibility/"null forms" conditions

  bearing on nonlinearities, at resonances

- *Laser-plasma interactions.*

  Euler (fluid) - Maxwell (electromagnetic waves).

[Germain-Masmoudi, 2014]
[Guo-Pausader-Ionescu, 2016]
} global existence results based on analysis of space-time resonances

Theorem. Two-fluid Euler-Maxwell with small parameter $\varepsilon > 0$:

- there exists approximate solutions u_a,
- there exists exact solutions u,

such that

- $\big\|(u - u_a)_{|t=0}\big\| = \varepsilon^K$ $\quad K > 0$ arbitrarily large,
- $\big\|(u - u_a)_{|t=O(\sqrt{\varepsilon}|\ln \varepsilon|)}\big\| \geq \varepsilon^{K'}$ $\quad K' > 0$ arbitrarily small.

$\|\cdot\|$: pointwise (L^∞) or weighted Sobolev (H^s_ε) initially.

$\|\cdot\|$: pointwise at final observation time.

ε : wavelength of incident light.

This theorem: a justification of high-frequency Râman instabilities for laser-plasma interactions.

Instability of an ansatz.

The ansatz:

$$u_a = \sum_{\substack{0 \le j \le K_a \\ p \in \mathcal{H}_j}} \varepsilon^{j/2} e^{ip(k\cdot x - \omega t)/\varepsilon} u_{a,j,p}\left(t, x, \frac{y}{\sqrt{\varepsilon}}\right),$$

where $u_{a,j,p}$ is independent of ε, K_a is large, $\mathcal{H}_j \subset \mathbb{Z}$ is finite.

The wavenumber k is given and the frequency ω is to be determined.

We show that this ansatz is

- *consistent*: leads to a well-posed system of equations,

  some compatibility conditions/null forms do hold

- *unstable*: in the sense of the theorem.

  some compatibility conditions/null forms do NOT hold

Ansatz: $u_a = \sum_{j,p} \varepsilon^{j/2} e^{ip(k\cdot x - \omega t)/\varepsilon} u_{a,j,p}\left(t, x, \frac{y}{\sqrt{\varepsilon}}\right).$

Leading profile: $u_{a,0,\pm 1}$; first corrector: $u_{a,1,0}, \dots$

The Zakharov system.

Components of the leading profile and first corrector satisfy

$$(\mathrm{Z}) \quad \begin{cases} i(\partial_t + c(k)\partial_x)E - \Delta_y E = n\,E, \\ (\partial_t^2 - \Delta_y)n = \Delta_y\big(|E|^2\big). \end{cases}$$

- LWP results for regular Sobolev data: [Schochet-Weinstein 1986] [Ozawa-Tsutsumi 1992] [Linares-Ponce-Saut 2005].
- Ghost effect: E a component of $u_{a,0,1}$, n a component of $u_{a,1,0}$ [Aoki-Takata 2001; Sone 2002].

Upshot: *the Zakharov approximation to Euler-Maxwell is unstable.*

Râman and Brillouin instabilities.

Instability of an ansatz for Euler-Maxwell:

- $\left\|(u - u_a)_{|t=0}\right\| \leq \varepsilon^K$ K arbitrarily large,
- $\left\|(u - u_a)_{|t_\varepsilon}\right\| \geq \varepsilon^{K'}$ K' arbitrarily small, $t_\varepsilon = O(\sqrt{\varepsilon}|\ln \varepsilon|)$.

What is responsible for the instability?

- Not the degree of precision of the approximate solution u_a.

  When plugged into Euler-Maxwell, u_a yields an arbitrarily small remainder $O(\varepsilon^{K''})$, K'' arbitrarily large.

- Not the smoothness of u_a or u.

  Both are C^∞ in $[0, t_\varepsilon] \times \mathbb{R}^3$. The key initial perturbation $u_a(0) - u(0)$ is C_c^∞.

- Not the structure of two-fluid Euler-Maxwell for fixed $\varepsilon > 0$.

  It's symmetric hyperbolic.

Instability of the Zakharov approximation to Euler-Maxwell:

- $\left\|(u - u_a)_{|t=0}\right\| = \varepsilon^K$ K arbitrarily large,
- $\left\|(u - u_a)_{|t_\varepsilon}\right\| \geq \varepsilon^{K'}$ K' arbitrarily small, $t_\varepsilon = O(\sqrt{\varepsilon}|\ln \varepsilon|)$.

Where are the singularities?

- The initial data are high-frequency

$$u_a(0, x, y) = \Re e \left(e^{ikx/\varepsilon} a\left(x, \frac{y}{\sqrt{\varepsilon}}\right)\right).$$

- The Euler-Maxwell system depends on ε via $1/\varepsilon$.

  In particular contains a bilinear term with a $1/\sqrt{\varepsilon}$ prefactor.

  Caricature: $y' = (1/\sqrt{\varepsilon})y^2$.

Euler-Maxwell:

(a) Maxwell in vacuum:

$$\partial_t B + \nabla \times E = 0, \quad \partial_t E - \nabla \times B = 0,$$

or

$$\partial_t \begin{pmatrix} B \\ E \end{pmatrix} + \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix} \begin{pmatrix} B \\ E \end{pmatrix} = 0.$$

Symmetric hyperbolic structure:

$$\partial_t + \sum_{1\le j\le 3} A_j \partial_{x_j}, \qquad A_j \text{ symmetric.}$$

(b) Euler-Maxwell describes *light-matter interactions:* systems of the form

$$\partial_t + A_0 + \sum_{1\le j\le 3} A_j \partial_{x_j}, \qquad A_0 \text{ skew-symmetric, } A_j \text{ symmetric.}$$

Planes waves supported by a symmetric hyperbolic system:

Given $\partial_t + A_0 + \sum_{1\leq j\leq 3} A_j \partial_{x_j}$, given $\xi \in \mathbb{R}^3$:

looking for $\lambda \in \mathbb{R}$ and $\vec{a} \in \mathbb{R}^N$ such that

$$\Big(\partial_t + A_0 + \sum_{1\leq j\leq 3} A_j \partial_{x_j}\Big)\Big(e^{i(\xi\cdot x - \lambda t)}\vec{a}\Big) \equiv 0$$

or

$$\begin{cases} \det\Big(- i\lambda + A_0 + i \sum_{1\leq j\leq 3} A_j \xi_j \Big) = 0, \\ \qquad \vec{a} \text{ corresponding eigenvector} \end{cases}$$

Solutions $\lambda = \lambda(\xi)$ are locally defined, *not* 1-homogeneous

ε is wavelength

Fast space-time oscillations: typical frequencies are $1/\varepsilon$.

$$\Big(\partial_t + \frac{1}{\varepsilon}A_0 + \sum_{1\leq j\leq 3} A_j \partial_{x_j}\Big)\big(e^{i(\xi\cdot x - \lambda t)/\varepsilon}\vec{a}\big) \equiv 0$$

or

$$\begin{cases} \det\big(-i\lambda + A_0 + i\sum_{1\leq j\leq 3} A_j\xi_j\big) = 0, \\ \vec{a} \text{ corresponding eigenvector} \end{cases}$$

The high-frequency linear hyperbolic operator is

$$\partial_t + \frac{1}{\varepsilon}\Big(\underbrace{A_0 + \sum_{1\leq j\leq 3} A_j \varepsilon \partial_{x_j}}_{=:A(\varepsilon\partial_x)}\Big).$$

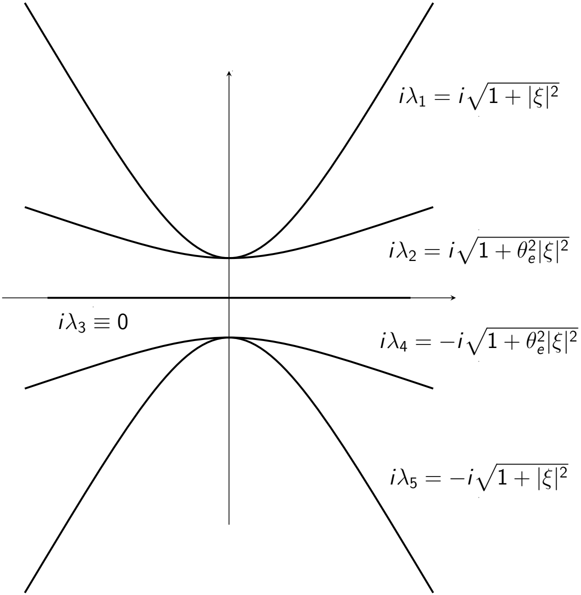
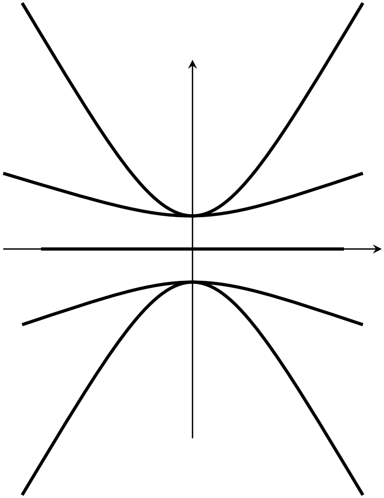


Figure: The characteristic variety for the Euler-Maxwell equations linearized at $u = 0$. The parameter θ_e is $\simeq 10^{-3}$.



Linearized Euler-Maxwell at $u = 0$:

- fast Klein-Gordon modes
  ($\leftrightarrow$ the laser)
- slow Klein-Gordon modes
  ($\leftrightarrow$ the electrons)
- slow acoustic modes
  ($\leftrightarrow$ the ions)

$$
(\text{EM}) \begin{cases}
\partial_t B + \nabla \times E = 0 \\
\partial_t E - \nabla \times B = \frac{1}{\varepsilon} e^{\sqrt{\varepsilon} n_e} v_e - \frac{1}{\theta_e \sqrt{\varepsilon}} e^{\sqrt{\varepsilon} n_i} v_i \\
\partial_t v_e + \sqrt{\varepsilon}\theta_e (v_e \cdot \nabla) v_e + \theta_e \nabla n_e = -\frac{1}{\varepsilon}(E + \sqrt{\varepsilon}\theta_e v_e \times B) \\
\partial_t n_e + \sqrt{\varepsilon}\theta_e (v_e \cdot \nabla) n_e + \theta_e \nabla \cdot v_e = 0 \\
\partial_t v_i + \varepsilon (v_i \cdot \nabla) v_i + \sqrt{\varepsilon} \nabla n_i = \frac{1}{\theta_e \sqrt{\varepsilon}}(E + \varepsilon v_i \times B) \\
\partial_t n_i + \varepsilon (v_i \cdot \nabla) n_i + \sqrt{\varepsilon} \nabla \cdot v_i = 0 \\
\nabla \cdot B = 0 \quad \nabla \cdot E = \frac{1}{\varepsilon \theta_e}(n_i - n_e)
\end{cases}
$$

that is

$$
\partial_t u + \underbrace{\frac{1}{\varepsilon}\left(A_0 + A(\sqrt{\varepsilon} u, \varepsilon \partial_x)\right)}_{\substack{\text{KG/KG/ac, frequencies } \sim 1/\varepsilon \\ \text{plus convection}}} u = \frac{1}{\sqrt{\varepsilon}} B(u,u) + \text{h.o.t.}
$$

Nonlinear instability from spectral instability:

For s large enough : $u \in C^0([0, T_*(\varepsilon)], H^s(\mathbb{R}^3))$.

$$\dot{u} := \tilde{u} - \tilde{u}_a, \quad \tilde{v}(t,x,y) = v(t,x,\sqrt{\varepsilon}y).$$

Let $t_*(\varepsilon)$ be s.t. $\mathcal{F}L^1$ are controlled ($\leq \varepsilon^{K'}$) on $[0, t_*(\varepsilon)]$:

$$t_*(\varepsilon) \geq T_\varepsilon \sqrt{\varepsilon} |\ln \varepsilon|, \quad T_\varepsilon = \frac{K - K'}{C} - \frac{C' \ln |\ln \varepsilon| + \ln C}{\gamma} \sqrt{\varepsilon}.$$

With $s - s_1$ large enough,

$$\|\dot{u}(t)\|_{\varepsilon, s_1} \lesssim \varepsilon^K |\ln \varepsilon|^\star e^{t\gamma/\sqrt{\varepsilon}} \text{ on } [0, t_*(\varepsilon)].$$

With $u_{in} = \sum_{q \in \mathcal{H}} e^{iq\theta} \mathrm{op}_\varepsilon(H_q)\dot{u}$:

$$\|u_{in}(T_\varepsilon \sqrt{\varepsilon} |\ln \varepsilon|, \cdot)\|_{L^\infty} \geq C\varepsilon^{K'},$$

$$\|u_{in}\|_{L^\infty} \lesssim |\dot{u}|_{L^\infty} \big(1 + |\ln \varepsilon| + |\ln \|\dot{u}\|_{\varepsilon, s_1}|\big).$$

First viewpoint on resonances:

resonances are stationary points for a relevant phase

$$\partial_t u + \underbrace{\frac{1}{\varepsilon}\big(A_0 + A(\cancel{\sqrt{\varepsilon} u}, \varepsilon\partial_x)\big)}_{\text{KG/KG/ac}} u = \frac{1}{\sqrt{\varepsilon}} B(u,u) + \cancel{\text{h.o.t.}}$$

Linearized equations around the WKB solution u_a :

$$\partial_t u + \frac{1}{\varepsilon}\underbrace{\overbrace{\big(A_0 + A(0, \varepsilon\partial_x)\big)}^{\text{linear hyperbolic}}}_{=:\mathcal{A}} u = \underbrace{\frac{1}{\sqrt{\varepsilon}}\overbrace{B(u_a, u) + B(u, u_a)}^{=:B(u_a)u}}_{\text{large linear source}}.$$

Implicit representation:

$$u = e^{t\mathcal{A}/\varepsilon} u(0) + \frac{1}{\sqrt{\varepsilon}} \int_0^t e^{(t-t')\mathcal{A}/\varepsilon} B(u_a(t'))u(t')\, dt'.$$

For very small time:

$$u(t) \simeq e^{t\mathcal{A}/\varepsilon} u(0),$$

hence key is bound for

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t e^{(t-t')\mathcal{A}/\varepsilon} B(u_a(t')) e^{t'\mathcal{A}/\varepsilon} u(0)\, dt'$$

which takes the form

$$\frac{1}{\sqrt{\varepsilon}} \int_0^t \exp\left(\frac{it'\Phi(\varepsilon D_x)}{\varepsilon} \right) f(t')\, dt'$$
$$= \frac{1}{\sqrt{\varepsilon}} \int_0^t \int_{\mathbb{R}^3_\xi} \exp\left(ix \cdot \xi + \frac{it'\Phi(\varepsilon\xi)}{\varepsilon} \right) \hat{f}(t', \xi)\, dt'\, d\xi.$$

Oscillatory integrals: $J = \frac{1}{\sqrt{\varepsilon}} \int_0^t \exp\left(\frac{it'\Phi(\varepsilon D_x)}{\varepsilon} \right) f(t')\, dt'$.

WKB datum: $u_a(0,x) = \Re e \left(e^{ik\cdot x/\varepsilon} a(x) \right)$.

Characteristic frequencies λ solve $\det\left(-i\lambda + A_0 + i\sum_j A_j \xi_j \right) = 0$

Non-stationary phase argument: away from the *resonant set*

$$\{\xi \in \mathbb{R}^3,\ \Phi(\xi) = 0\}$$

integrate by parts in time to find $J = O(\sqrt{\varepsilon})$.

The *phase* takes the form

$$\Phi(\xi) = \lambda_j(\xi + k) - \omega - \lambda_{j'}(\xi),$$

where

- k is the initial wavenumber and ω an associated characteristic frequency
- λ_j and $\lambda_{j'}$ are characteristic frequencies

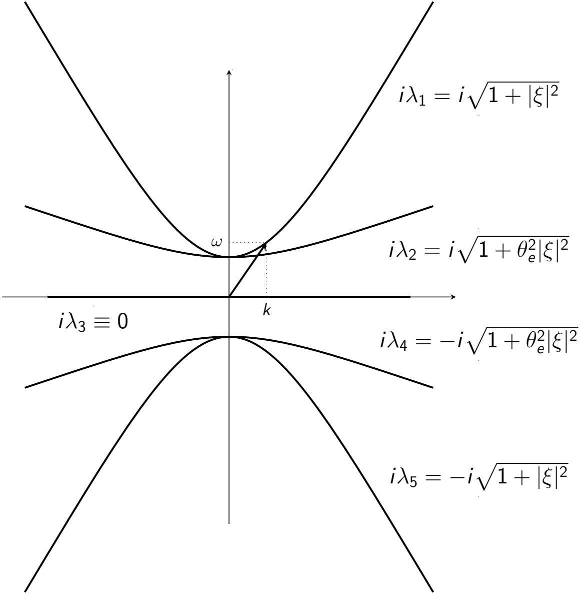


Figure: The characteristic variety, i.e. solutions $\xi \to \lambda(\xi)$ of $\det\left(-i\lambda + A_0 + i\sum_j A_j\xi_j\right) = 0$ and the fundamental phase (ω, k).

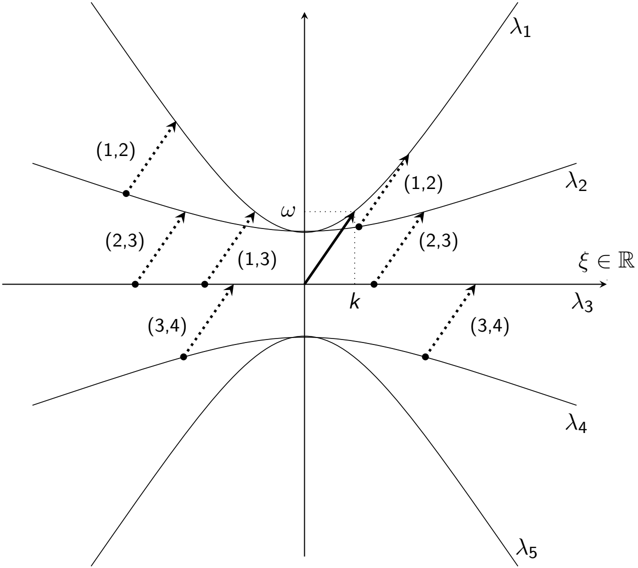


Figure: Examples of resonances, ie zeros of $\lambda_j(k+\cdot) - \omega - \lambda_{j'}(\cdot)$.

Oscillatory integrals: $\frac{1}{\sqrt{\varepsilon}} \int_0^t \exp\left(\frac{it'\Phi(\varepsilon D_x)}{\varepsilon}\right) f(t')\,dt'$.

Away from the resonant set $\{\xi \in \mathbb{R}^3,\ \Phi(\xi) = 0\}$ integrate by parts in time.

What do we do close to the resonant set?

- With some luck $\hat{f} = 0$ (a *compatibility* condition)
  [Klainerman: null conditions; Joly-Métivier-Rauch: transparency]

- Or: integrate by parts in ξ whenever possible!
  [Germain-Masmoudi-Shatah: *space-time* resonances]
  Space-time resonances are frequencies ξ that belong to

$$\{\Phi = 0\} \cap \{\partial_\xi \Phi = 0\}.$$

Second viewpoint on resonances:

resonances are small divisors in a homological equation [Poincaré]

$$\partial_t u + \frac{1}{\varepsilon}\mathbf{A}u = \frac{1}{\sqrt{\varepsilon}}\mathbf{B}u$$

two distinct scales: ε, ξ

$\mathbf{A}$: order one (as a differential operator). $\mathbf{B}$: order zero.

Looking for $\mathbf{Q}$ of order -1 such that

$$v = (\mathrm{Id} + \sqrt{\varepsilon}\mathbf{Q})^{-1} u$$

solves a simpler equation. We find

$$\partial_t v + \frac{1}{\varepsilon}\mathbf{A}v = \frac{1}{\sqrt{\varepsilon}}\big(\mathbf{B} - [\mathbf{A}, \mathbf{Q}]\big) v + \text{l.o.t.}$$

Homological equation:

$$\mathbf{B} - [\mathbf{A}, \mathbf{Q}] = 0 \qquad (\text{or } = \sqrt{\varepsilon}\tilde{\mathbf{Q}}, \text{ with } \tilde{\mathbf{Q}} \text{ order } 0).$$

$$\partial_t u + \frac{1}{\varepsilon}\mathbf{A}u = \frac{1}{\sqrt{\varepsilon}}\mathbf{B}u \qquad \text{change of variable: } v = (\mathrm{Id} + \sqrt{\varepsilon}\mathbf{Q})^{-1}u \text{ solves}$$

$$\partial_t v + \frac{1}{\varepsilon}\mathbf{A}v = \frac{1}{\sqrt{\varepsilon}}\big(\mathbf{B} - [\mathbf{A}, \mathbf{Q}]\big)v + \text{l.o.t.}$$

Homological equation:

$$\mathbf{B} - [\mathbf{A}, \mathbf{Q}] = 0$$

takes the form

$$(\lambda_j(\xi + k) - \lambda_{j'}(\xi) - \omega)Q_{jj'} = \Pi_j(\xi + k)B(u_a)\Pi_{j'}(\xi)$$

with $Q_{jj'}$ one entry in the matrix-valued pseudo-differential symbol $\mathbf{Q}(\varepsilon, t, x, \xi)$.

Away from the zeros of $\lambda_j(\cdot + k) - \lambda_{j'}(\cdot) - \omega$, we *remove* $\mathbf{B}$ from the right-hand side of the equation.

This focuses the analysis in the frequency space to a neighboorhood of the resonant set.

Third viewpoint on resonances:

the resonant set is the locus of weak hyperbolicity for an equivalent linear operator

meaning that there is a change of variable such that

$$\partial_t u + \frac{1}{\varepsilon}\mathbf{A}u = \frac{1}{\sqrt{\varepsilon}}\mathbf{B}u$$

transforms into a collection of systems of the form

$$\partial_t v + \frac{1}{\varepsilon}\begin{pmatrix} i(\lambda_j(\varepsilon\partial_x + k) - \omega) & 0 \\ 0 & i\lambda_{j'}(\varepsilon\partial_x) \end{pmatrix} v = \frac{1}{\sqrt{\varepsilon}}\begin{pmatrix} 0 & b^+ \\ b^- & 0 \end{pmatrix} v$$

and bounds for u imply bounds for v and conversely.

Existence of such a change of variable depends heavily on the structure of the resonant set.

From $\quad \partial_t u + \frac{1}{\varepsilon}\mathbf{A}u = \frac{1}{\sqrt{\varepsilon}}\mathbf{B}u \quad$ to

$$\partial_t v + \frac{1}{\varepsilon}\begin{pmatrix} i(\lambda_j(\varepsilon\partial_x + k) - \omega) & 0 \\ 0 & i\lambda_{j'}(\varepsilon\partial_x) \end{pmatrix} v = \frac{1}{\sqrt{\varepsilon}}\begin{pmatrix} 0 & b^+ \\ b^- & 0 \end{pmatrix} v$$

The eigenvalues of

$$\frac{1}{\varepsilon}\begin{pmatrix} i(\lambda_j(\xi + k) - \omega) & 0 \\ 0 & i\lambda_{j'}(\xi) \end{pmatrix} - \frac{1}{\sqrt{\varepsilon}}\begin{pmatrix} 0 & b^+ \\ b^- & 0 \end{pmatrix}$$

are

$$\frac{1}{2\varepsilon}\Big(i(\lambda_j(\xi + k) - \omega + \lambda_{j'}(\xi)) \pm \Big(4\varepsilon b^+ b^- - (\lambda_j(\xi + k) - \omega - \lambda_{j'}(\xi))^2\Big)^{1/2}\Big).$$

- Far from the zeros of $\lambda_j(\cdot + k) - \omega - \lambda_{j'}(\cdot)$, the spectrum is purely imaginary.
- At a zero of $\lambda_j(\cdot + k) - \omega - \lambda_{j'}(\cdot)$, the lower-order perturbation terms $\sqrt{\varepsilon}b^{\pm}$ may cause the spectrum to bifurcate away from the imaginary axis.

Euler-Maxwell: $\partial_t u + \frac{1}{\varepsilon}\big(A_0 + A(\sqrt{\varepsilon}u, \varepsilon\partial_x, \sqrt{\varepsilon}\partial_y)\big)u = \frac{1}{\sqrt{\varepsilon}}B(u,u).$

WKB approximate solution $u_a = \sum_{j,p} \varepsilon^{j/2} e^{ip(k\cdot x - \omega t)/\varepsilon} u_{a,j,p}\left(t, x, \frac{y}{\sqrt{\varepsilon}}\right).$

Linearized E-M around WKB: $\partial_t u + \frac{1}{\varepsilon}\big(A_0 + A(\sqrt{\varepsilon}u_a, \varepsilon\partial_x, \sqrt{\varepsilon}\partial_y)\big)u = \frac{1}{\sqrt{\varepsilon}}B(u_a)u.$

Goal: bounds in time $O(\sqrt{\varepsilon}|\ln\varepsilon|)$.

Goal is to get a grasp on the linear perturbed operator

$$\frac{1}{\varepsilon}(A_0 + A(\sqrt{\varepsilon}u_a, \varepsilon\partial_x, \sqrt{\varepsilon}\partial_y)) - \frac{1}{\sqrt{\varepsilon}}B(u_a).$$

in time $O(\sqrt{\varepsilon}|\ln\varepsilon|)$.

Issues:

- singular prefactor $1/\sqrt{\varepsilon}$
- singularity of the WKB solution: $u_a \simeq e^{i(k\cdot x - \omega t)/\varepsilon} u_{a,0,1}(t, x, y/\sqrt{\varepsilon})$

Linearized E-M around WKB: $\partial_t u + \frac{1}{\varepsilon}\big(A_0 + A(\sqrt{\varepsilon} u_a, \varepsilon\partial_x, \sqrt{\varepsilon}\partial_y)\big) u = \frac{1}{\sqrt{\varepsilon}} B(u_a) u.$

The WKB approximate solution $u_a \simeq e^{i(k\cdot x - \omega t)/\varepsilon} u_{a,0,1}(t, x, y/\sqrt{\varepsilon})$

Issues: fast oscillations, singular WKB profile, large $1/\sqrt{\varepsilon}$ prefactor

---

~~Our approach:~~ spectrum of $\frac{1}{\varepsilon}(A_0 + A(\sqrt{\varepsilon} u_a, \varepsilon\partial_x)) - \frac{1}{\sqrt{\varepsilon}} B(u_a).$

Spectral approach to instability problems in fluid mechanics: [Grenier 2000, Gérard-Varet, Dormy 2005]

Spectral approach to stability of traveling waves: [Sattinger 1976, ... Liu, Serre, Zumbrun]

Our approach: spectrum of $\frac{1}{\varepsilon}(A_0 + A(\sqrt{\varepsilon} u_a, \boxed{i\varepsilon\xi, i\sqrt{\varepsilon}\eta})) - \frac{1}{\sqrt{\varepsilon}} B(u_a).$

We study spectra of *symbols* rather than operators.

Symbols are (x, y, ξ, η)-dependent matrices. Hence spectral problem in finite dimensions.

The symbolic flow method reduces a spectral problem:

$$\mathrm{sp} A(u_s(t,x,y), \varepsilon\partial_x, \sqrt{\varepsilon}\partial_y) : \quad ?? \tag{1}$$

into a spectral problem *in finite dimensions:*

$$\mathrm{sp} A(u_s(t,x,y), i\xi, i\eta) \tag{2}$$

Instead of having to compute (1) we compute (2) *and* from (2) deduce

how?? trivial

estimates for the solution to

$$\partial_t u + \frac{1}{\varepsilon} A(u_s(t,x,y), \varepsilon\partial_x, \sqrt{\varepsilon}\partial_y) u = 0.$$

Limitations:

- short time $O(\sqrt{\varepsilon}|\ln \varepsilon|)$
- order-zero operators

[Lu, T 2015][T, 2016][Lerner, Nguyen, T 2016]

The symbolic flow method: the solution to

$$\partial_t u + \frac{1}{\varepsilon}\mathrm{op}_\varepsilon \underbrace{\begin{pmatrix} i\chi(\lambda_j(\cdot + k) - \omega) & 0 \\ 0 & i\chi\lambda_{j'} \end{pmatrix}}_{=:i\mathbf{A}} u = \frac{1}{\sqrt{\varepsilon}}\mathrm{op}_\varepsilon \underbrace{\begin{pmatrix} 0 & \chi b^+ \\ \chi b^- & 0 \end{pmatrix}}_{=:\mathbf{B}} u$$

is given by

$$u \simeq \mathrm{op}_\varepsilon(S(0;t))u(0), \qquad t \leq T\sqrt{\varepsilon}|\ln\varepsilon|,$$

where S solves

$$\partial_t S + \frac{i}{\varepsilon}\mathbf{A}S + \frac{1}{\sqrt{\varepsilon}}\partial_\eta \mathbf{A}\cdot\partial_y S = \frac{1}{\sqrt{\varepsilon}}\mathbf{B}S, \qquad S(\tau;\tau) = \mathrm{Id}.$$

$$\mathrm{op}_\varepsilon(a)f = \int_{\mathbb{R}^3} e^{ix\xi + iy\cdot\eta} a(x, y, \varepsilon\xi, \sqrt{\varepsilon}\eta)\hat{f}(\xi,\eta)d\xi d\eta, \qquad (\xi,\eta)\in\mathbb{R}\times\mathbb{R}^2.$$

χ : smooth frequency truncation around the bounded resonant set.

The auxiliary partial differential equation for the symbolic flow:

$S(\tau; t, x, y, \xi, \eta)$ solves

$$\partial_t S + \frac{i}{\varepsilon}\mathbf{A}S + \frac{1}{\sqrt{\varepsilon}}\partial_\eta \mathbf{A}\cdot\partial_y S = \frac{1}{\sqrt{\varepsilon}}\mathbf{B}S, \qquad S(\tau;\tau) = \mathrm{Id}.$$

Symbolic analysis:

$$\mathcal{M}(\varepsilon, t, x, \xi, \eta; y, \hat{y}) := \frac{1}{\varepsilon}\Big(i\mathbf{A} - \sqrt{\varepsilon}\mathbf{B} + \sqrt{\varepsilon}i\hat{y}\cdot\partial_\eta \mathbf{A}\Big)$$
$$= \frac{1}{\varepsilon}\begin{pmatrix} i(\lambda_1(\xi+k,\eta) - \omega + \sqrt{\varepsilon}\hat{y}\cdot\partial_\eta\lambda_1(\xi+k,\eta)) & \sqrt{\varepsilon}\chi b^+(x,y,\xi,\eta) \\ \sqrt{\varepsilon}\chi b^-(x,y,\xi,\eta) & i(\lambda_2 + \sqrt{\varepsilon}\hat{y}\cdot\partial_\eta\lambda_2) \end{pmatrix},$$

with eigenvalues $2\lambda^\pm_{\mathcal{M}} = \operatorname{tr}\mathcal{M} \pm \delta^{1/2}$, with $(\varphi := \lambda_1(\cdot + k) - \omega - \lambda_2)$

$$\delta := -\varphi^2 + 2\sqrt{\varepsilon}\varphi\hat{y}\cdot\partial_\eta\varphi - \varepsilon(\hat{y}\cdot\partial_\eta\varphi)^2 + 4\varepsilon b^+ b^-$$

Symbolic analysis for the operator in the PDE for the symbolic flow:

$$\text{real eigenvalues} \iff \delta > 0 \iff \quad \text{instability.}$$

With φ the resonant phase:

$$\delta := -\varphi^2 + 2\sqrt{\varepsilon}\varphi\hat{y} \cdot \partial_\eta \varphi - \varepsilon(\hat{y} \cdot \partial_\eta \varphi)^2 + 4\varepsilon b^+ b^-$$

- $\varphi(\xi, \eta) \neq 0 \implies \delta < 0$: no instability far from resonances.
- $\varphi(\xi, \eta) = 0$ and $\partial_\eta \varphi(\xi, \eta) \neq 0$: a resonance which is not a space-time resonance. By spatial decay of the WKB profile, y is small hence $\hat{y}$ is large. Hence $-\varepsilon(\hat{y} \cdot \partial_\eta \varphi)^2$ dominates and $\delta < 0$.
- $\varphi(\xi, \eta) = 0$, $\partial_\eta \varphi(\xi, \eta) = 0$: instability if $b^+ b^- > 0$.