Wave-Structure interactions: Oscillating water column in shallow water

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Outline

- Motivation and background
- Two transmission problems
- Well-posedness
- Numerical scheme and discretizations

Motivation: Oscillating water column (OWC)

OWC installed in 1990 at Trivandrum, India.

OWC installed in Australia, about 2005.

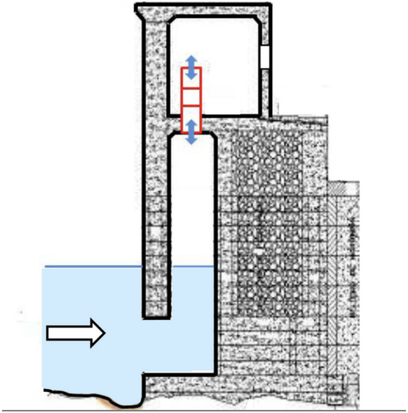
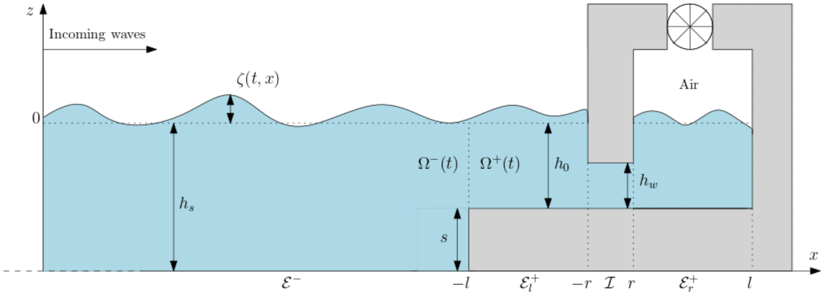


Figure: Taken from Falcao, Henriques, Renewable Energy, 2015.

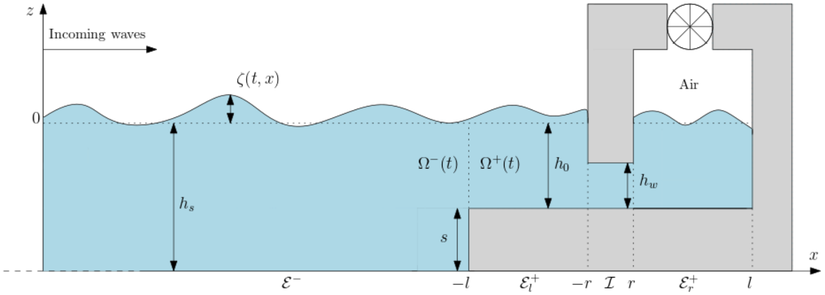
Reduce to two transmission problems



Notations

- $\zeta(x,t)$ is the surface elevation around the rest state;
- $h(x,t)$ is the fluid height (at rest h_s before the step, h_0 after the step);
- $q(x,t)$ is the horizontal discharge ($q = \int_{-h_{bot}}^{\zeta} udz = h\overline{u}$);
- $\underline{P}(x,t)$ is the surface pressure.

Mathematical configuration



Constraints and unknowns

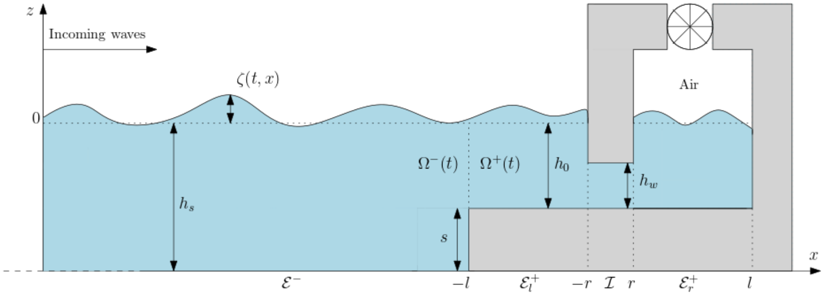
- Exterior domain: $\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}_l^+ \cup \mathcal{E}_r^+$ (domain before the step, before the structure and inside the chamber):

  $\underline{P} = P_{\mathrm{atm}}$ in $\mathcal{E}^- \cup \mathcal{E}_l^+$; $\underline{P} = P_{\mathrm{atm}} + P_{ch}(t)$ in $\mathcal{E}_r^+$; ζ is unknown in $\mathcal{E}$

- Interior domain $\mathcal{I} = (l_0 - r, l_0 + r)$: (under the structure):

  $\underline{P}$ is unknown $\zeta = \zeta_w$ (constant in t and x).

Mathematical configuration



Previous results

- D. Lannes, On the dynamics of floating structures, 2017;
- E. Godlewski, M. Parisot, J. Sainte-Marie and F. Wahl, relaxation of the constraint, 2018;
- T. Iguchi and D. Lannes, 1D NSW equations, 2019;
- E. Bocchi, for the 2D-radial NSW equations, 2019;
- D. Maity, J. San Martin, T. Takahashi, M. Tucsnak, visous 1D NSW, 2019;
- D. Lannes, L. Weynans, boundary conditions for Boussinesq, 2019;
- D. Bresch, D. Lannes, G. Mérivier, for the 1D-Boussinesq and fixed solid, 2020.

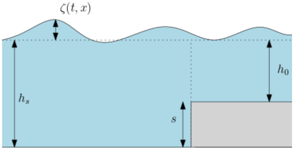
....

Derivation of the Model

Step 1 : Transmission problem near the Step

The motion of wave is described by the 1D shallow water equations :

$$\mathcal{E}^- : \begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} \right) + gh\partial_x \zeta = \underbrace{-\dfrac{1}{\rho} h \partial_x \underline{P}_{\text{atm}}}_{=0} \\ h = h_s + \zeta, \quad \underline{P} = P_{\text{atm}} \end{cases}$$



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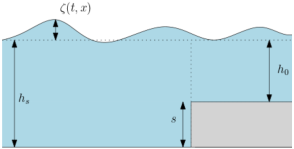

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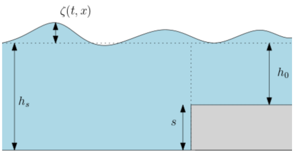

$\Updownarrow$ First transmission problem: $\zeta|_{x=0^-} = \zeta|_{x=0^+}, \; q|_{x=0^-} = q|_{x=0^+}$

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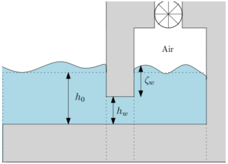

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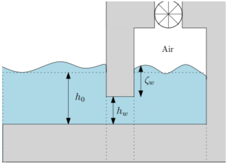

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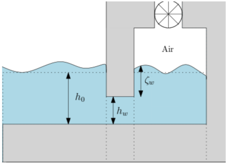

Coupling conditions :

Continuity of q

$q(t, l_0 \pm r) = q_i(t, l_0 \pm r)$

$\zeta_w = \zeta_i \rightsquigarrow \partial_t \zeta_i = 0 \rightsquigarrow q_i(t, x) = q_i(t)$

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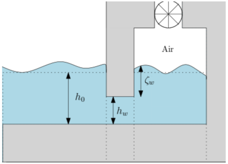

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$\rightsquigarrow$ first transmission condition :

$$q_{|x=l_0+r} - q_{|x=l_0-r} := [\![q]\!] = 0$$

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Step 3: Derive the second transmission condition near the Structure

Local energy :

Exterior : $\partial_t \mathfrak{e}_{\rm ext} + \partial_x \mathfrak{f}_{\rm ext} = P_{\rm air} \partial_x q;$

- $\mathfrak{e}_{\rm ext} = \rho \frac{q^2}{2h} + g\rho \frac{\zeta^2}{2}$ and $\mathfrak{f}_{\rm ext} = q\left(\rho \frac{q^2}{2h^2} + g\rho\zeta + P_{air}\right).$

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Interior : $\partial_t \mathfrak{e}_{\rm int} + \partial_x \mathfrak{f}_{\rm int} = 0$.

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Global energy :

$$\text{Exterior : } \partial_t \int_{\mathcal{E}} \mathfrak{e}_{\rm ext} + \partial_x \int_{\mathcal{E}} \mathfrak{f}_{\rm ext} = \int_{\mathcal{E}} P_{\rm air}\partial_x q; \quad \text{Interior : } \partial_t \int_{\mathcal{I}} \mathfrak{e}_{\rm int} + \partial_x \int_{\mathcal{I}} \mathfrak{f}_{\rm int} = 0.$$

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Exterior : $\partial_t \int_{\mathcal{E}} \mathfrak{e}_{\mathrm{ext}} + \partial_x \int_{\mathcal{E}} \mathfrak{f}_{\mathrm{ext}} = \int_{\mathcal{E}} P_{\mathrm{air}} \partial_x q$; Interior : $\partial_t \int_{\mathcal{I}} \mathfrak{e}_{\mathrm{int}} + \partial_x \int_{\mathcal{I}} \mathfrak{f}_{\mathrm{int}} = 0$.

We then have

$$\partial_t \left(\int_{\mathcal{E}} \mathfrak{e}_{\mathrm{ext}} + \int_{\mathcal{I}} \mathfrak{e}_{\mathrm{int}} + E_{\mathrm{sol}} \right) + [\![\mathfrak{f}_{\mathrm{int}}]\!] - [\![\mathfrak{f}_{\mathrm{ext}}]\!] = -[\![P_{\mathrm{air}} q]\!] = -q_i P_{\mathrm{ch}}.$$

Remarque : $[\![P_{\mathrm{air}}]\!] = (P_{\mathrm{atm}} + P_{\mathrm{ch}}) - P_{\mathrm{atm}} = P_{\mathrm{ch}}$

Step 3: Derive the second transmission condition near the Structure

The perturbation $P_{\mathrm{ch}}(t)$ satisfies the ODE (without damping) [1]

$$\frac{d}{dt}P_{\mathrm{ch}} = \gamma_1 q_i \quad \Longrightarrow \quad \frac{1}{2\gamma_1}\frac{d}{dt}P_{\mathrm{ch}}^2 = q_i P_{\mathrm{ch}}$$

where γ_1 is a known physical parameter.

[1] The ODE of the Pressure is from ocean engineering literature : Dimakopoulos-Cooker-Bruce 2017, Falcão-Henriques-Gato 2016...

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2. $[\![\mathfrak{f}_{\mathrm{ext}}]\!] = \left[\!\!\left[q(\rho\frac{q^2}{2h^2} + g\rho\zeta + P_{\mathrm{air}}) \right]\!\!\right]$

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We derive that

$$\frac{P_{\mathrm{ch}}(t)}{\rho} + \left[\!\!\left[\frac{q^2}{2h^2} + g\zeta \right]\!\!\right] + \frac{2r}{h_w}\frac{d}{dt}q_i = 0$$

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Two transmission problems

$$\text{In } (-l, 0): \quad \begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{h}\right) + gh\partial_x \zeta = 0 \\ h = h_s + \zeta, \ \underline{P} = P_{\mathrm{atm}} \end{cases}$$

First transmission problem $\updownarrow \zeta_{|_{x=0-}} = \zeta_{|_{x=0+}}, q_{|_{x=0-}} = q_{|_{x=0+}}$

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$$\text{In } (0, l_0 - r): \quad \begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{h}\right) + gh\partial_x \zeta = 0 \\ h = h_0 + \zeta, \ \underline{P} = P_{\mathrm{atm}} \end{cases}$$

Second trans. prob. $\updownarrow [\![q]\!] = 0, \quad \langle q \rangle = q_i,$ where q_i, P_{ch} *satisfy ODE*

$$\text{In } (l_0 + r, l_1): \quad \begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x \left(\frac{q^2}{h}\right) + gh\partial_x \zeta = 0 \\ h = h_0 + \zeta, \ \underline{P} = P_{\mathrm{atm}} + P_{\mathrm{ch}}(t) \end{cases}$$

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Transmission problems $\rightsquigarrow$ IBVP with semi-linear boundary conditions ?

Reduction to an IBVP with semi-linear boundary conditions

$$q_{|_{x=l_0+r}} - q_{|_{x=l_0-r}} := [\![q]\!] = 0$$

$$\frac{1}{2}\left(q_{|_{x=l_0-r}} + q_{|_{x=l_0+r}}\right) := \langle q \rangle = q_i$$

$$\Longrightarrow$$

$$\mathcal{M}^+ U_{|_{x=l_0+r}} - \mathcal{M}^- U_{|_{x=l_0-r}} = V(G(t))$$

$\mathcal{M}^\pm$ are 2×2 constant matrices

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Reformulate the system as

$$\begin{cases} \partial_t U + A(U)\partial_x U = 0 & \text{in} \quad (0,T) \times \mathbb{R}^-, \\ \partial_t U + A(U)\partial_x U = 0 & \text{in} \quad (0,T) \times \mathbb{R}^+, \\ U_{|_{t=0}} = U_0(x) & \text{on} \quad \mathbb{R}_- \cup \mathbb{R}_+, \\ \mathcal{M}^+ U_{|_{x=0^+}} - \mathcal{M}^- U_{|_{x=0^-}} = V(G(t)) & \text{on} \quad (0,T). \end{cases}$$

where U, U_0 are $\mathbb{R}^2$-valued functions, $A(U)$ is a 2×2 real-valued matrix, $V(\cdot)$ is a $\mathbb{R}^2$-valued given function with $V(0) = 0$ and G is a $\mathbb{R}^2$-valued function satisfying the ODE.

Quasilinear hyperbolic IBVP (with some diagonalizability) with semi-linear boundary conditions

$$(PDE)\begin{cases}\partial_t u + \mathcal{A}(u)\partial_x u = 0\\ u_{|_{t=0}} = u_0(x)\\ \mathcal{M}u_{|_{x=0}} = V(G(t))\end{cases} \quad \text{with} \quad (ODE)\begin{cases}\dot{G} = \Theta\left(G, u_{|_{x=0}}\right),\\ G(0) = G_0.\end{cases}$$

Well-posedness

Hyperbolic linear initial boundary value problems

Let us first consider a linear system of the form (with constant matrix)

$$\begin{cases} \partial_t u + A\partial_x u = 0 & \text{in} \quad (0,T)\times\mathbb{R}_+, \\ u_{|_{t=0}} = u_0 & \text{on} \quad \mathbb{R}_+, \\ \nu \cdot u_{|_{x=0}} = G(t) & \text{on} \quad (0,T), \end{cases}$$

- A: a 2×2 constant matrix with eigenvalues $\pm\lambda_\pm$ and eigenvectors $\mathbf{e}_\pm$;
- $\nu \in \mathbb{R}^2$ and Kreiss-Lopatinskiĭ condition $\nu \cdot \mathbf{e}_+ \neq 0$;

Hyperbolic linear initial boundary value problems

Let us first consider a linear system of the form (with constant matrix)

$$\begin{cases} \partial_t u + A\partial_x u = 0 & \text{in} \quad (0,T)\times\mathbb{R}_+, \\ u_{|_{t=0}} = u_0 & \text{on} \quad \mathbb{R}_+, \\ \nu\cdot u_{|_{x=0}} = G(t) & \text{on} \quad (0,T), \end{cases}$$

- A: a 2×2 constant matrix with eigenvalues $\pm\lambda_\pm$ and eigenvectors $\mathbf{e}_\pm$;
- $\nu\in\mathbb{R}^2$ and Kreiss-Lopatinskiĭ condition $\nu\cdot\mathbf{e}_+\neq 0$; **Why ?**

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Consider a general boundary condition

$$\mathcal{M}u_{|_{x=0}} = G(t) \quad \text{on} \quad (0,T).$$

where $\mathcal{M}$ is a $m\times 2$ real-valued matrix.
Applying the Laplace transform to the system

$$s\widehat{u} + A\partial_x\widehat{u} = 0, \qquad \mathcal{M}\widehat{u} = \widehat{G} \quad \text{at} \quad x=0.$$

Then the general solution is

$$\widehat{u}(s,x) = \widehat{c_+}(s)\exp(-s\lambda_+^{-1}x)\mathbf{e}_+ + \widehat{c_-}(s)\exp(s\lambda_-^{-1}x)\mathbf{e}_-.$$

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The boundary condition becomes $(\mathcal{M}\mathbf{e}_+)\widehat{c_+} = \widehat{G} \rightsquigarrow m = 1 \rightsquigarrow \mathcal{M} = \nu^T$, so

$$\widehat{c_+} = \frac{1}{\nu\cdot\mathbf{e}_+}\widehat{G} \quad \rightsquigarrow \quad \nu\cdot\mathbf{e}_+ \neq 0$$

Kreiss symmetrizer: a matrix S such that SA is symmetric and

- there exist constants $c_1, C_1 > 0$ such that

$$c_1|v|^2 \leq v^T S(t,x) v \leq C_1|v|^2;$$

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L^2 **energy estimate:** multiplying equation $\partial_t u + A\partial_x u = 0$ by S and taking the $L^2((0,t) \times \mathbb{R})$ scalar product with u, after integration by parts,

$$(Su(t), u(t))_{L^2} - \int_0^t (SAu \cdot u)_{|x=0} = (Su_0, u_0)_{L^2}.$$

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Note that the trace $-(SAu\cdot u)_{|_{x=0}}$ cannot be controlled by the L^2-norm of u. Kreiss symmetrizer is a symmetrizer such that the term $-(SAu\cdot u)_{|_{x=0}}$ has good sign and the trace $|u_{|_{x=0}}|_{L^2(0,t)}$ can be controlled up to terms that depends only on $\nu \cdot u_{|_{x=0}} = G$.

$$\|u(t)\|_{L^2} + |u_{|_{x=0}}|_{L^2(0,t)} \leq C(K_0)e^{C(K)t}\left(\|u(0)\|_{L^2} + \|G\|_{L^2(0,t)}\right).$$

Compatibility conditions

There are m necessary conditions for the existence of a solution of regularity $C^{m-1}([0, T] \times \mathbb{R}_+)$

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$$u_{0,k+1} = u_{k+1|t=0} = -\partial_x u_{0,k} \quad (1)$$

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Definition: Let $m\geq 1$. We say that the data $u_0\in H^m(\mathbb{R}_+)$ and $G\in H^m(0,T)$ for the IBVP satisfy the compatibility condition at order $m-1$ if $\{u_{0,j}\}_{j=0}^m$ defined in (1) satisfy (2) for $k=0,1,\cdots,m-1$.

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- $\mathcal{A}(t,x)$ is a 4×4 matrix
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[2] T Iguchi, D Lannes. *Hyperbolic free boundary problems and applications to wave-structure interactions*, 2021

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- u_0, G satisfy the compatibility conditions.
- Well-posedness [2] : *a priori* L^2 estimate; high-order estimates; existence and uniqueness; existence of a Kreiss symmetrizer.

[2] T Iguchi, D Lannes. *Hyperbolic free boundary problems and applications to wave-structure interactions*, 2021

Hyperbolic quasilinear initial boundary value problems

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Sketch of proof of well-posedness.

Step 1. Choice of an iterative scheme.

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Step 2. High-norm boundedness.

$$\|u^n\|_{\mathbb{W}^m(T_1)} + |u^n_{|_{x=0}}|_{m,T_1} \le M, \quad \forall n \in \mathbb{N}.$$

Step 3. Low-norm convergence. One proves that the sequence u^n is convergent in L^2 and that the limit is in space $\mathbb{W}^m := \bigcap_{j=0}^m C^j([0,T]; H^{m-j}(\mathbb{R}_+))$, endowed with the norm

$$\|u\|_{\mathbb{W}^m(T)} = \sup_{t\in[0,T]} \sum_{j=0}^m \|\partial_t^j u(t)\|_{H^{m-j}(\mathbb{R}_+)}.$$

Hyperbolic IBVP with semi-linear boundary conditions

$$(PDE)\begin{cases}\partial_t u + \mathcal{A}(u)\partial_x u = 0\\ u_{|_{t=0}} = u_0(x)\\ \mathcal{M}u_{|_{x=0}} = G(t)\end{cases} \quad \text{with} \quad (ODE)\begin{cases}\dot{G} = \Theta\left(G, u_{|_{x=0}}\right),\\ G(0) = G_0.\end{cases}$$

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Theorem (2021)

Let $T > 0$ and $m \geq 2$ be an integer. Suppose that $u_0 \in H^m(\mathbb{R})$ and $G_0 \in H^m(0, T)$ satisfy the compatibility conditions up to order $m-1$, then there exist $0 < T_1 < T$ and a unique solution (u, G) to (PDE-ODE) with

$$u \in \bigcap_{j=0}^{m} C^j([0, T]; H^{m-j}(\mathbb{R}_+)) := \mathbb{W}^m(T) \quad \textit{and} \quad G \in H^{m+1}(0, T).$$

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Key ingredients needed:

- Kreiss-Lopatinskiĭ conditions (Lopatinskiĭ matrix is invertible), compatibility conditions;
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- Estimates for G and the trace $u_{|_{x=0}}$

Goal: boundedness of (u^n, G^n) in $\mathbb{W}^m(T) \times H^{m+1}(0,T)$;
convergence of (u^n, G^n) in $\mathbb{W}^{m-1}(T) \times H^{m-1}(0,T)$.

Recall that

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- Iterative scheme of ODE

$$\dot{G}^{n+1} = \Theta\left(G^n, u^n_{|x=0}\right), \quad G^{n+1}(0) = G_0.$$

- Uniform bounded. By $G^{n+1}(t) = G^{n+1}_{|t=0} + \int_0^t \Theta\left(G^n(s), u^n(s)\right)\, ds$, we have

$$\begin{aligned}\|G^{n+1}\|_{H^m(0,T_1)} &\lesssim \sqrt{T_1}\|\Theta(G^n, u^n_{|x=0})\|_{H^m(0,T_1)}\\ &\lesssim \sqrt{T_1} C_\Theta \left(\|G^n\|_{H^m(0,T_1)} + |u^n_{|x=0}|_{m,T_1} \right)\end{aligned}$$

- Convergence.

$$\begin{aligned}&\|G^{n+1} - G^n\|_{H^{m-1}(0,T_1)}\\ \lesssim &\sqrt{T_1} C_\Theta \left(\|G^n - G^{n-1}\|_{H^{m-1}(0,T_1)} + |(u^n - u^{n-1})_{|x=0}|_{m-1,T_1} \right).\end{aligned}$$

Numerical scheme and discretization of BCs

Riemann invariants

1D Nonlinear shallow water equations in a compact form are given by

$$\partial_t U + A(U)\partial_x U = 0 \quad \text{with } U = \begin{pmatrix} \zeta \\ q \end{pmatrix}, \; A(U) = \begin{pmatrix} 0 & 1 \\ gh - \frac{q^2}{h^2} & \frac{2q}{h} \end{pmatrix}$$

with eigenvalues

$$\lambda_+(U) = \sqrt{gh} + \frac{q}{h} > 0, \quad -\lambda_-(U) = -\sqrt{gh} + \frac{q}{h}$$

Taking the scalar product of the eq. with the associated eigenvectors, we obtain two transport equations

$$\partial_t R(U) + \lambda_+(U)\partial_x R(U) = 0, \quad \partial_t L(U) - \lambda_-(U)\partial_x L(U) = 0,$$

for the right-going and left-going Riemann invariants respectively given by

$$R(U) = 2(\sqrt{gh} - \sqrt{gh_0}) + \frac{q}{h}, \quad L(U) = 2(\sqrt{gh} - \sqrt{gh_0}) - \frac{q}{h}.$$

Discretization of the Model

Let us first rewrite the shallow water equations in a more compact form by introducing $U = (\zeta, q)^T$:

$$\partial_t U + \partial_x (F(U)) = 0,$$

with

$$F(U) = (q, \frac{1}{2} g (h^2 - h_0^2) + \frac{q^2}{h})^T,$$

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Then the Lax-Friedrichs scheme for solving the above partial differential equation is given by:

$$\frac{U_i^{n+1} - \frac{1}{2}(U_{i+1}^n + U_{i-1}^n)}{\Delta t} + \frac{F(U_{i+1}^n) - F(U_{i-1}^n)}{2\,\Delta x} = 0$$

which implies

$$U_i^{n+1} = \frac{1}{2}(U_{i+1}^n + U_{i-1}^n) - \frac{\Delta t}{2\,\Delta x}(F(U_{i+1}^n) - F(U_{i-1}^n))$$

Entry condition

- Surface elevation ζ is given by $\zeta(x = -l, t^n) = f(t^n) \leftarrow$ cosine
- Horizontal discharge q can be derived by Left Riemann invariant L :

$$q = h(2(\sqrt{gh} - \sqrt{gh_s}) - L)$$

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After discretization, we have

$$q^n|_{x=-l} = (h_s + f(t^n))(2(\sqrt{g(h_s + f(t^n))} - \sqrt{gh_s}) - \overbrace{L^n|_{x=-l}}^{?}).$$

Entry condition

- Surface elevation ζ is given by $\zeta(x=-l,t^n)=f(t^n)\leftarrow$ cosine
- Horizontal discharge q can be derived by Left Riemann invariant L :

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Discretizing the characteristic equation of L at $x_0=-l$, we have

$$\frac{L_0^n-L_0^{n-1}}{\delta_t}-\lambda_-\frac{L_1^{n-1}-L_0^{n-1}}{\delta_x}=0.$$

Thus, $L^n|_{x=-l}$ can be determined by

$$L_0^n = (1-\lambda_-\frac{\delta_t}{\delta_x})L_0^{n-1}+\lambda_-\frac{\delta_t}{\delta_x}L_1^{n-1}.$$

Discretization of discontinuous topography

- Continuity of ζ and q: $\zeta^l|_{x=0} = \zeta^r|_{x=0}$; $q^l|_{x=0} = q^r|_{x=0}$

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Using Riemann invariants, we find two expressions of q describing $q^l|_{x=0}$ and $q^r|_{x=0}$, respectively,

$$\begin{cases} q^l|_{x=0} = (h_s + \zeta^l|_{x=0})\Big(R^l|_{x=0} - 2(\sqrt{g(h_s + \zeta^l|_{x=0})} - \sqrt{g h_s})\Big) \\ q^r|_{x=0} = (h_0 + \zeta^r|_{x=0})\Big(2(\sqrt{g(h_0 + \zeta^r|_{x=0})} - \sqrt{gh_0}) - L^r|_{x=0}\Big) \end{cases}$$

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Set $X = \sqrt{h_0 + \zeta^l|_{x=0}}$, we derive a 5-th order polynomial equation

$$AX^5 + BX^4 + CX^3 + DX^2 + E = 0$$

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where the coefficients depend on $R^l|_{x=0}$ and $L^r|_{x=0}$, which can be determined by their characteristic equations :

$$(R^l)_0^n = \left(1 - \lambda_+^l \frac{\delta_t}{\delta_x}\right)(R^l)_0^{n-1} + \lambda_+^l \frac{\delta_t}{\delta_x}(R^l)_{-1}^{n-1},$$

$$(L^r)_0^n = \left(1 - \lambda_-^r \frac{\delta_t}{\delta_x}\right)(L^r)_0^{n-1} + \lambda_-^r \frac{\delta_t}{\delta_x}(L^r)_{+1}^{n-1}$$

Transmission conditions across the structure

- $[\![q]\!] = 0 \quad \rightsquigarrow \quad q^l|_{l_0 - r} = q_i = q^r|_{l_0 + r}.$

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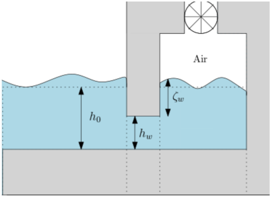
- We discretize the second transmission condition

$$-\alpha\frac{dq_i}{dt} = \frac{P_{\mathrm{ch}}(t)}{\rho} + \left[\!\!\left[\frac{q^2}{2h^2} + g\zeta\right]\!\!\right].$$

$$(q)^n_{l_0-r} = (q)^{n-1}_{l_0-r} - \frac{\delta t}{\alpha}\left(\frac{\left((q^l)^{n-1}_{l_0+r}\right)^2}{2\left(h_0+(\zeta^l)^{n-1}_{l_0+r}\right)^2} - \frac{\left((q^r)^{n-1}_{l_0-r}\right)^2}{2\left(h_0+(\zeta^r)^{n-1}_{l_0-r}\right)^2}\right)$$
$$-\frac{\delta t}{\alpha}\left(g(\zeta^l)^{n-1}_{l_0+r} - g(\zeta^r)^{n-1}_{l_0-r}\right) - \frac{\delta t}{\rho\alpha}P^{n-1}_{\mathrm{ch}}.$$

Wall condition at the end of the chamber

- $q(x = l_1, t) = 0$
- using the right Riemann invariant we have



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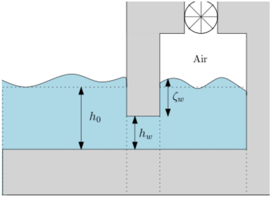

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After discretization, we have

$$\zeta^n_{|_{x=l_1}} = \frac{1}{g}\left(\frac{R^n_{|_{x=l_1}}}{2} + \sqrt{gh_0}\right)^2 - h_0$$

where $R^n_{|_{x=l_1}}$ is obtained by the discretized characteristic equations as before.

Simulation

Wave energy converter

Thanks for your attention !