Journées EDP de l'IECL 2022

Camouflage d'obstacles dans des guides d'ondes acoustiques au moyen de ligaments fins résonants

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Coll. with J. Heleine[1], S.A. Nazarov[2].

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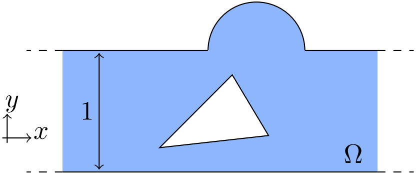
Nancy, 28/03/2022

► We consider the propagation of waves in a 2D acoustic waveguide with an obstacle (also relevant in optics, microwaves, water-waves theory,...).



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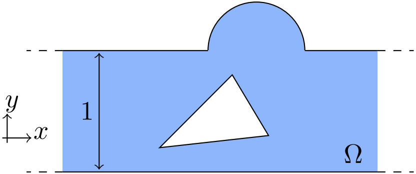

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► We fix $k \in (0;\pi)$ so that only the plane waves $e^{\pm ikx}$ can propagate.

► The scattering of these waves leads us to consider the solutions of $(\mathscr{P})$ with the decomposition

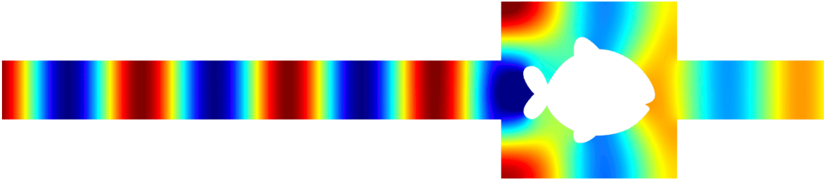
$$u_+ = \left| \begin{array}{l} e^{ikx} + R_+ \, e^{-ikx} + \dots \\ \qquad T \;\; e^{+ikx} + \dots \end{array} \right. \qquad u_- = \left| \begin{array}{ll} \qquad T \;\; e^{-ikx} + \dots & x \to -\infty \\ e^{-ikx} + R_- \, e^{+ikx} + \dots & x \to +\infty \end{array} \right.$$

$R_\pm$, $T \in \mathbb{C}$ are the scattering coefficients , the $\dots$ are expon. decaying terms.

- We have the relations of conservation of energy $|R_{\pm}|^2 + |T|^2 = 1$.

- Without obstacle, $u_+ = e^{ikx}$ so that $(R_+, T) = (0, 1)$.

- With an obstacle, in general $(R_+, T) \neq (0, 1)$.



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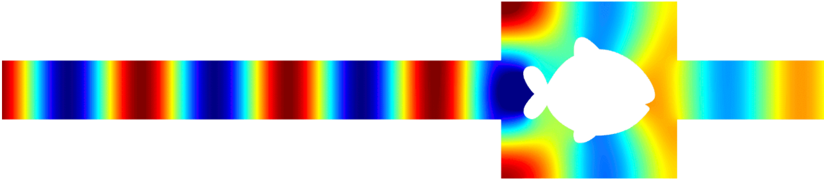

Goal of the talk

We wish to slightly perturb the walls of the guide to obtain $R_{\pm} = 0$, $T = 1$ in the new geometry (as if there were no obstacle) $\Rightarrow$ cloaking at "infinity".

Introduction 3/3

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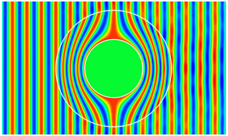

Remark 1: Different from the usual cloaking picture (Pendry *et al.* 06, Leonhardt 06, Greenleaf *et al.* 09).
$\rightarrow$ Less ambitious but doable without fancy materials (and relevant in practice).

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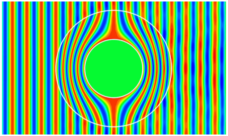

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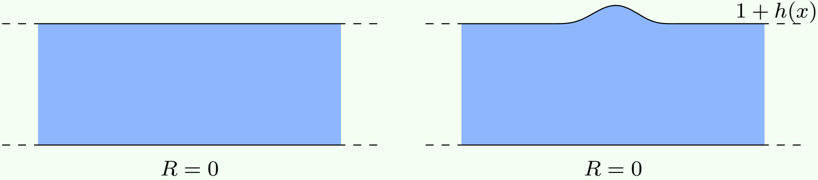
Remark 2: Different from the perturbative techniques we have used in the past based on variants of the implicit functions theorem.



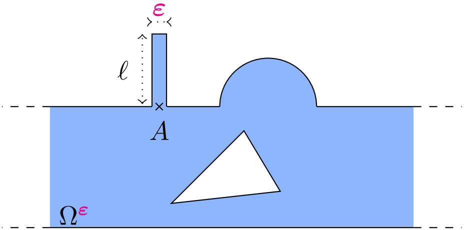
Here the (big) obstacle is given, we want to compensate its scattering.

Outline of the talk

1. Asymptotic analysis in presence of thin resonators
2. Almost zero reflection
3. Cloaking
4. Mode converter

Setting

Main ingredient of our approach: outer resonators of width $\varepsilon \ll 1$.



$$(\mathscr{P}^\varepsilon)\ \left|\ \begin{array}{rl} \Delta u + k^2 u = 0 & \text{in } \Omega^\varepsilon, \\ \partial_n u = 0 & \text{on } \partial\Omega^\varepsilon \end{array}\right.$$

- In this geometry, we have the scattering solutions

$$u_+^\varepsilon = \left| \begin{array}{r} e^{ikx} + R_+^\varepsilon\, e^{-ikx} + \dots \\ T^\varepsilon\, e^{+ikx} + \dots \end{array}\right. \qquad u_-^\varepsilon = \left| \begin{array}{r} T^\varepsilon\, e^{-ikx} + \dots \\ e^{-ikx} + R_-^\varepsilon\, e^{+ikx} + \dots \end{array}\right. \qquad \begin{array}{l} x \to -\infty \\ x \to +\infty \end{array}$$

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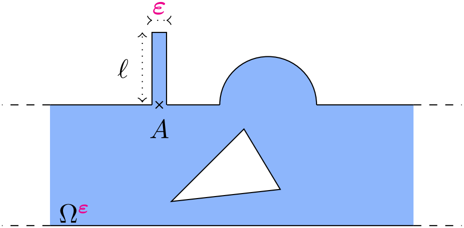

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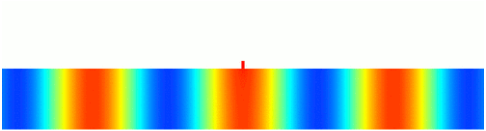
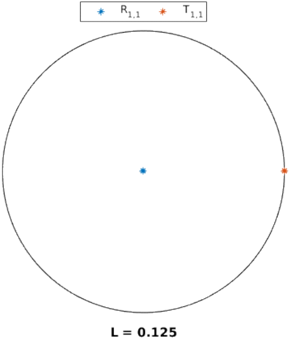
In general, the thin ligament has only a weak influence on the scattering coefficients: $R_\pm^\varepsilon \approx R_\pm$, $T^\varepsilon \approx T$. But not always ...

Numerical experiment

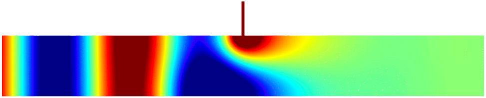
- We vary the length of the ligament:

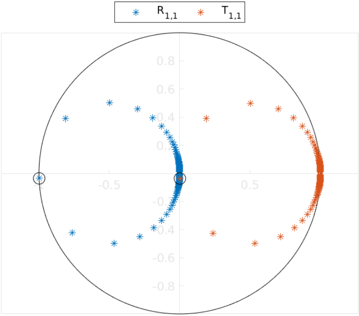




Numerical experiment

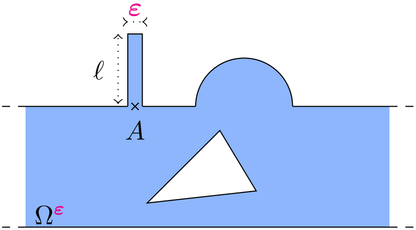
- For one particular length of the ligament, we get a standing mode (zero transmission):





Asymptotic analysis

To understand the phenomenon, we compute an asymptotic expansion of u_+^ε, R_+^ε, T^ε as $\varepsilon \to 0$.



$$(\mathscr{P}^\varepsilon)\left|\begin{array}{rl} \Delta u_+^\varepsilon + k^2 u_+^\varepsilon = 0 & \text{in } \Omega^\varepsilon, \\ \partial_n u_+^\varepsilon = 0 & \text{on } \partial\Omega^\varepsilon \end{array}\right.$$

$$u_+^\varepsilon = \left|\begin{array}{l} e^{ikx} + R_+^\varepsilon\, e^{-ikx} + \dots \\ \qquad\quad T^\varepsilon\, e^{+ikx} + \dots \end{array}\right.$$

► To proceed we use techniques of matched asymptotic expansions (see Beale 73, Gadyl'shin 93, Kozlov et al. 94, Nazarov 96, Maz'ya et al. 00, Joly & Tordeux 06, Lin & Zhang 17, 18, ...).

Asymptotic analysis

- We work with the outer expansions

$$\begin{aligned} u_+^\varepsilon(x,y) &= u^0(x,y) + \dots && \text{in } \Omega, \\ u_+^\varepsilon(x,y) &= \varepsilon^{-1} v^{-1}(y) + v^0(y) + \dots && \text{in the resonator.} \end{aligned}$$

- Considering the restriction of $(\mathscr{P}^\varepsilon)$ to the thin resonator, when ε tends to zero, we find that v^{-1} must solve the homogeneous 1D problem

$$(\mathscr{P}_{1\mathrm{D}}) \left| \begin{array}{ll} \partial_y^2 v + k^2 v = 0 & \text{in } (1; 1+\ell) \\ v(1) = \partial_y v(1+\ell) = 0. & \end{array} \right.$$

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- We denote by ℓ_{res} (**resonance lengths**) the values of ℓ, given by

$$\ell_{\mathrm{res}} := \pi(m + 1/2)/k, \qquad m \in \mathbb{N},$$

such that $(\mathscr{P}_{1\mathrm{D}})$ admits the non zero solution $v(y) = \sin(k(y-1))$.

Asymptotic analysis – Non resonant case

- Assume that $\ell \neq \ell_{\text{res}}$. Then we find $v^{-1} = 0$ and when $\varepsilon \to 0$, we get

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\begin{aligned}
&u^\varepsilon_\pm(x,y) = u_\pm + o(1) && \text{in } \Omega,\\
&u^\varepsilon_\pm(x,y) = u_\pm(A)\, v_0(y) + o(1) && \text{in the resonator},\\
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Here $v_0(y) = \cos(k(y-1) + \tan(k(y-\ell)\sin(k(y-1)$.

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The thin resonator has no influence at order ε^0.

$\rightarrow$ **Not interesting for our purpose** because we want $\left|\begin{array}{l} R_\pm^\varepsilon = 0 + \dots \\ T^\varepsilon = 1 + \dots \end{array}\right.$

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- Inner expansion. Set $\xi = \varepsilon^{-1}(\text{x} - A)$ (stretched coordinates). Since

$$(\Delta_{\text{x}} + k^2)u_+^{\varepsilon}(\varepsilon^{-1}(\text{x} - A)) = \varepsilon^{-2}\Delta_{\xi}u^{\varepsilon}(\xi) + \dots,$$

when $\varepsilon \to 0$, we are led to study the problem

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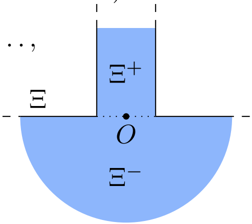

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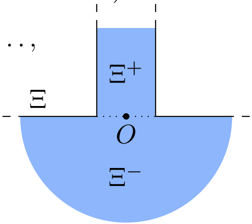

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- Problem $(\star)$ admits a solution Y^1 (up to a constant) with the expansion

$$Y^1(\xi) = \begin{cases} \xi_y + C_\Xi + O(e^{-\pi\xi_y}) & \text{as } \xi_y \to +\infty, \quad \xi \in \Xi^+ \\ \dfrac{1}{\pi}\ln\dfrac{1}{|\xi|} + O\Big(\dfrac{1}{|\xi|}\Big) & \text{as } |\xi| \to +\infty, \quad \xi \in \Xi^-. \end{cases}$$

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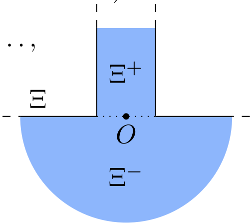

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$$u_+^\varepsilon(x) = C^A\,Y^1(\xi) + c^A + \dots \qquad (c^A,\ C^A \text{ constants to determine}).$$

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Since at A, the Taylor formula gives

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we take $C^A = ak$.

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- In the ansatz $u_+^\varepsilon = u^0 + \dots$ in Ω, we deduce that we must take

$$u^0 = u_+ + ak\gamma$$

where γ is the outgoing Green function such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n\gamma = \delta_A \text{ on } \partial\Omega. \end{array} \right.$

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- Matching the constant behaviour in the resonator, we obtain

$$v^0(1) = u_+(A) + ak(\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi).$$

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$$\left|\begin{array}{ll}\partial_y^2 v^0 + k^2 v^0 = 0 & \text{in } (1; 1+\ell)\\ v^0(1) = u_+(A) + ak(\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi), & \partial_y v^0(1+\ell) = 0.\end{array}\right.$$

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- This is a Fredholm problem with a non zero kernel. A solution exists iff the **compatibility condition** is satisfied. This sets

$$ak = -\frac{u_+(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi}$$

and **ends the calculus of the first terms**.

Asymptotic analysis – Resonant case

- Finally for $\ell = \ell_{\text{res}}$, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + ak\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

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$$R_+^\varepsilon = R_+ + iau_+(A)/2 + o(1), \qquad T^\varepsilon = T + iau_-(A)/2 + o(1).$$

Here γ is the outgoing Green function such that $\left| \begin{array}{l} \Delta\gamma + k^2\gamma = 0 \text{ in } \Omega \\ \partial_n\gamma = \delta_A \text{ on } \partial\Omega \end{array} \right.$ and

$$ak = -\frac{u_+(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi}.$$

This time the thin resonator has an influence at order ε^0

Asymptotic analysis – Resonant case

- Similarly for $\ell = \ell_{\text{res}} + \varepsilon\eta$ with $\eta \in \mathbb{R}$ fixed, by modifying only the last step with the compatibility relation, when $\varepsilon \to 0$, we obtain

$$u_+^\varepsilon(x,y) = u_+(x,y) + a(\eta)k\gamma(x,y) + o(1) \quad \text{in } \Omega,$$

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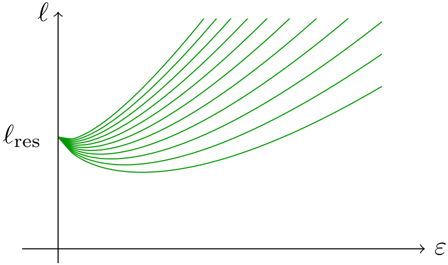
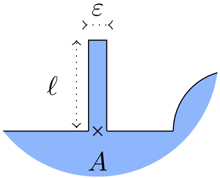
This time the thin resonator has an influence at order ε^0 and it depends on the choice of η!

Asymptotic analysis – Resonant case

- Below, for several $\eta \in \mathbb{R}$, we display the paths

$$\{(\varepsilon, \ell_{\text{res}} + \varepsilon(\eta - \pi^{-1}|\ln \varepsilon|)),\ \varepsilon > 0\} \subset \mathbb{R}^2.$$





According to η, the limit of the scattering coefficients along the path as $\varepsilon \to 0^+$ is different.

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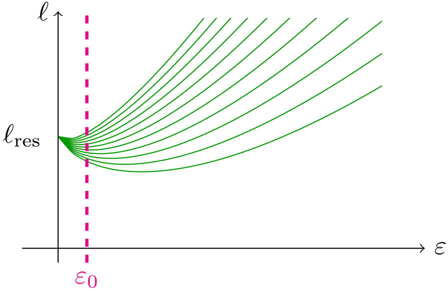

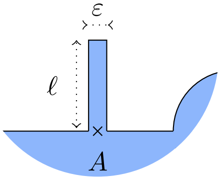

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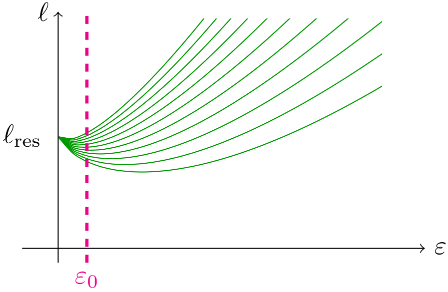
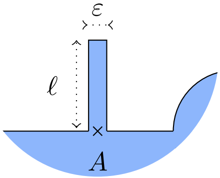




According to η, the limit of the scattering coefficients along the path as $\varepsilon \to 0^+$ is different.

- For a fixed small ε_0, the scattering coefficients have a rapid variation for ℓ varying in a neighbourhood of the resonance length.

$\rightarrow$ **This is exactly what we observed in the numerics.**

1 Asymptotic analysis in presence of thin resonators

2 Almost zero reflection

Varying the length of the ligament around the resonant lengths, we can get a rapid and large variation of the scattering coefficients.

$\rightarrow$ How to use that to get zero reflection ?

3 Cloaking

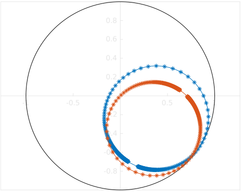
4 Mode converter

Almost zero reflection

- We have found $R_+^\varepsilon = R_+^0(\eta) + o(1), \quad T^\varepsilon = T^0(\eta) + o(1) \quad$ with

$$R_+^0(\eta) = R_+ + \frac{(2ik)^{-1}\, u_+(A)^2}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi + \eta}, \quad T^0(\eta) = T + \frac{(2ik)^{-1}\, u_+(A)u_-(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi + \eta}.$$

- Results on Möbius transform ($z \mapsto \frac{az+b}{cz+d}$) guarantee that $\{R_+^0(\eta)\,|\,\eta\in\mathbb{R}\}$, $\{T^0(\eta)\,|\,\eta\in\mathbb{R}\}$ are circles in $\mathbb{C}$.



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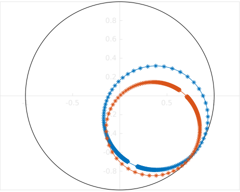

Asymptotically, when the length of the resonator is perturbed **around the resonance length**, R_+^ε, T^ε run on circles.

Almost zero reflection

- We have found $R_+^\varepsilon = {\color{blue}R_+^0(\eta)} + o(1), \quad T^\varepsilon = {\color{red}T^0(\eta)} + o(1)$ with

$$R_+^0(\eta) = R_+ + \frac{(2ik)^{-1}\, u_+(A)^2}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi + \eta}, \quad T^0(\eta) = T + \frac{(2ik)^{-1}\, u_+(A)u_-(A)}{\Gamma + \pi^{-1}\ln|\varepsilon| + C_\Xi + \eta}.$$

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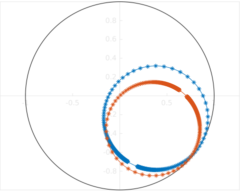
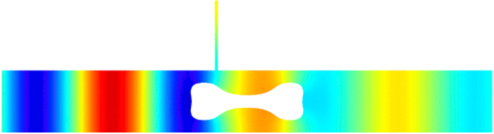
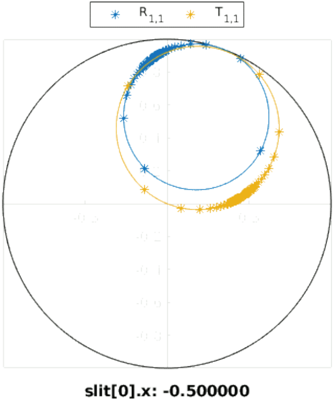


Asymptotically, when the length of the resonator is perturbed **around the resonance length**, R_+^ε, T^ε run on circles.

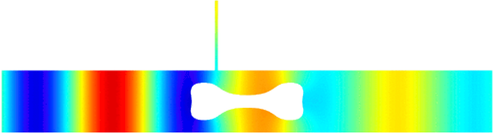
- Interestingly, the features of the circles depend on the position A of the ligament.

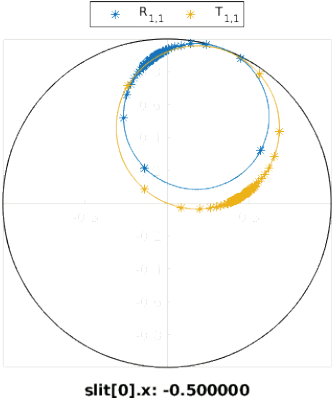
Almost zero reflection





Almost zero reflection

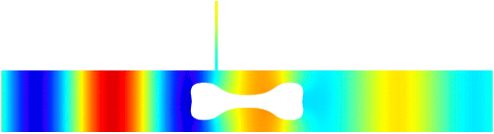


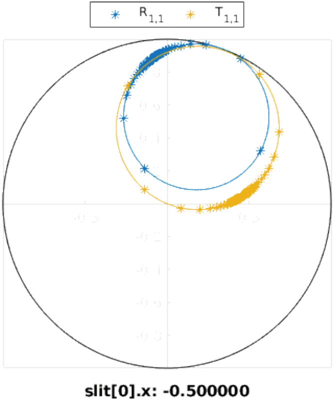


- Using the expansions of $u_{\pm}(A)$ far from the obstacle, one shows:

PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_{+}^{0}(\eta)\,|\,\eta \in \mathbb{R}\}$ passes **through zero**.

Almost zero reflection





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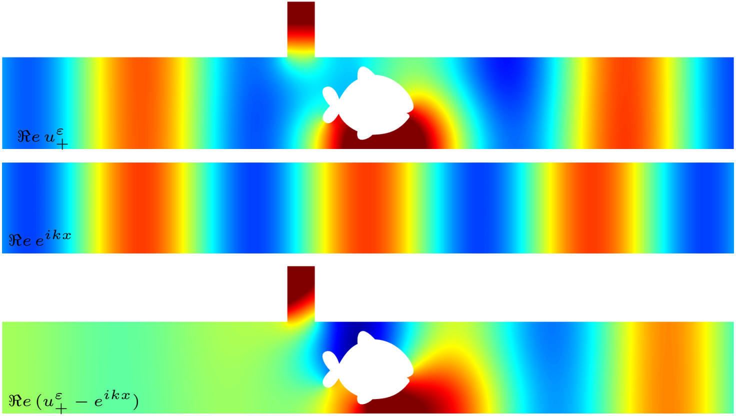
PROPOSITION: There are **positions of the resonator A** such that the circle $\{R_{+}^{0}(\eta)\,|\,\eta \in \mathbb{R}\}$ passes **through zero**. $\Rightarrow \exists$ situations s.t. $R_{+}^{\varepsilon} = 0 + o(1)$.

Almost zero reflection

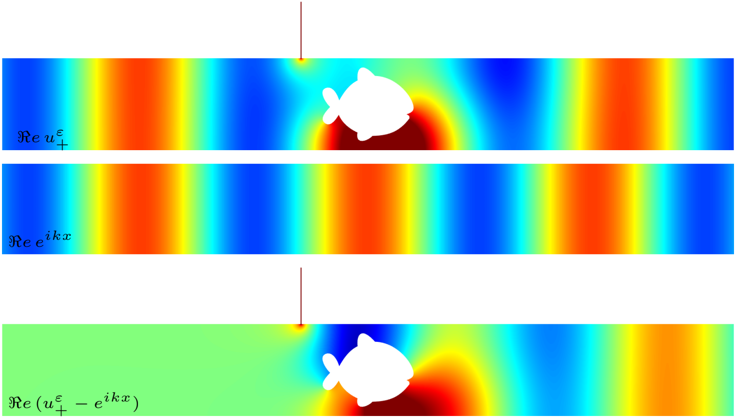
- Example of situation where we have almost zero reflection ($\boldsymbol{\varepsilon = 0.3}$).



Simulations realized with the `Freefem++` *library.*

Almost zero reflection

- Example of situation where we have almost zero reflection ($\boldsymbol{\varepsilon = 0.01}$).



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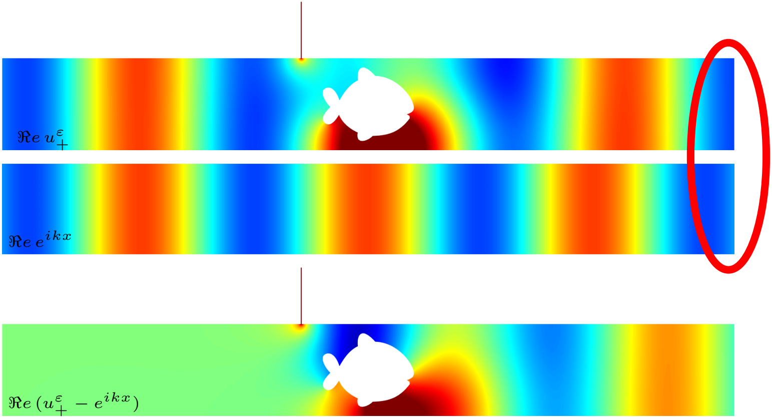


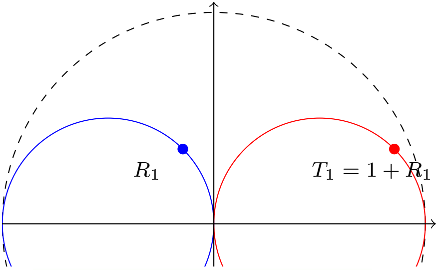
Simulations realized with the `Freefem++` *library.*

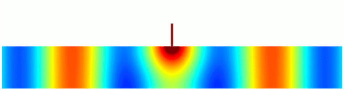
Conservation of energy guarantees that when $R_+^\varepsilon = 0$, $|T^\varepsilon| = 1$.
→ To cloak the object, it remains to compensate the phase shift!

Phase shifter

- Working with two resonators, we can create phase shifters, that is devices with almost zero reflection and any desired phase.

Scheme of the method:





Step 1: with one ligament, we get some R_1, T_1 as above.

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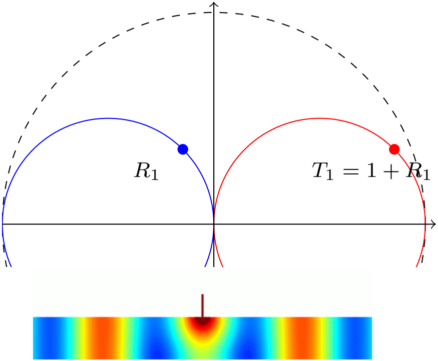

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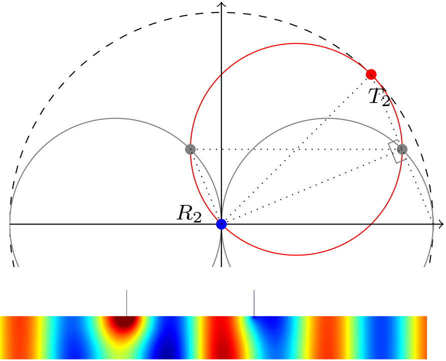

Step 2: adding a second ligament, we can get R_2, T_2 as above.

Phase shifter

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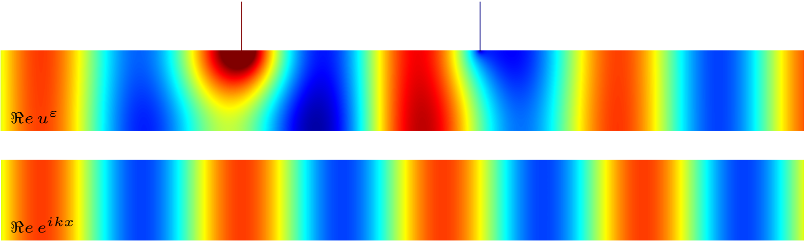


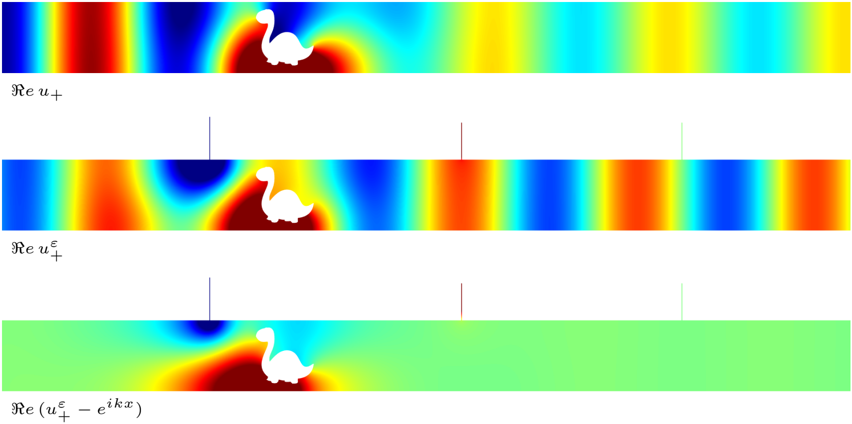
- Here the device is designed to obtain a phase shift approx. equal to $\pi/4$.

Cloaking with three resonators

- Now working in two steps, we can approximately cloak any object with three resonators:

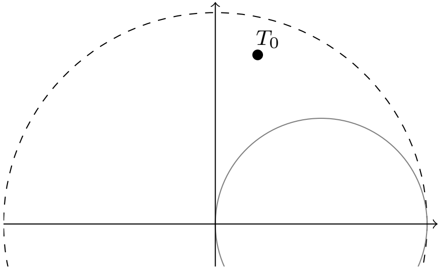
1) With one resonant ligament, first we get almost zero reflection;

2) With two additional resonant ligaments, we compensate the phase shift.



Cloaking with two resonators

- Working a bit more, one can show that two resonators are enough to cloak any object.



Step 1: add one ligament so that the corresponding transmission circle, which passes through zero and T_0, crosses $\mathscr{C}(1/2, 1/2) \setminus \{0\}$.

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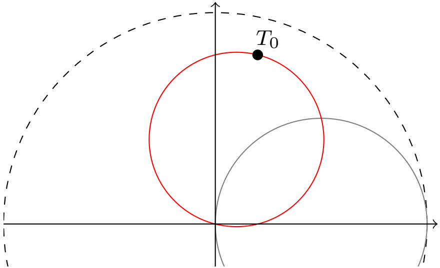

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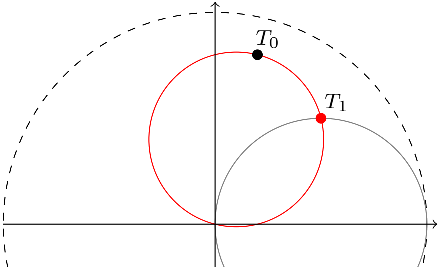

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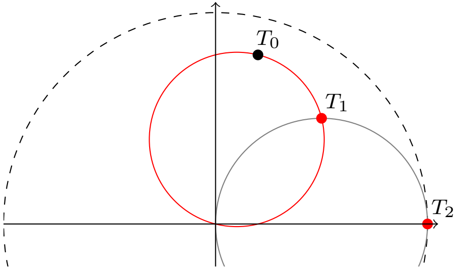

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Step 3: add a second ligament and tune its position as well as its length to get $T_2 = 1$ (this is doable because of the value of T_1).

Cloaking with two resonators

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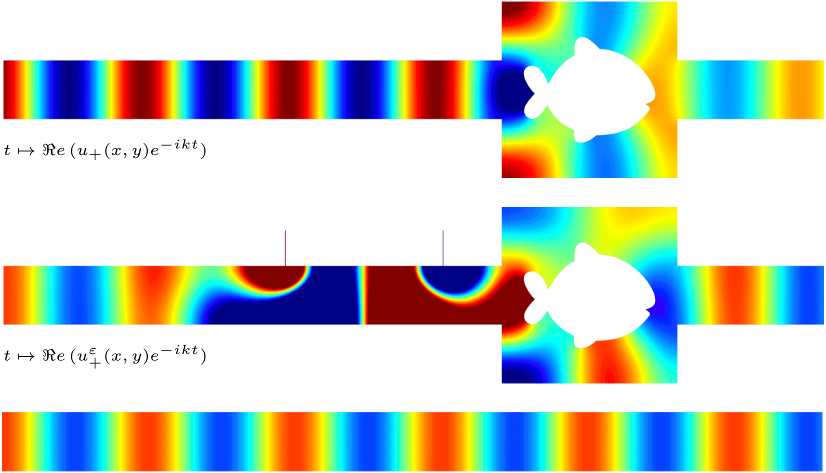


$t \mapsto \Re e\,(u_+(x,y)e^{-ikt})$

$t \mapsto \Re e\,(u_+^\varepsilon(x,y)e^{-ikt})$

$t \mapsto \Re e\,(e^{ik(x-t)})$

Mode converter - goal

- We work at higher k ($\in (\pi; 2\pi)$) so that two modes can propagate:

$$w_1^{\pm}(x,y) = e^{\pm i\beta_1 x}\varphi_1(y), \qquad w_2^{\pm}(x,y) = e^{\pm i\beta_2 x}\varphi_2(y).$$

- Now we have the two scattering solutions

$$u_1^{\varepsilon}(x,y) = \left| \begin{array}{ll} w_1^{+}(x+1/2,y) + \sum_{j=1}^{2} r_{1j}^{\varepsilon} w_j^{-}(x+1/2,y) + \dots & \text{on the left} \\ \sum_{j=1}^{2} t_{1j}^{\varepsilon} w_j^{+}(x-1/2,y) + \dots & \text{on the right} \end{array} \right.$$

$$u_2^{\varepsilon}(x,y) = \left| \begin{array}{ll} w_2^{+}(x+1/2,y) + \sum_{j=1}^{2} r_{2j}^{\varepsilon} w_j^{-}(x+1/2,y) + \dots & \text{on the left} \\ \sum_{j=1}^{2} t_{2j}^{\varepsilon} w_j^{+}(x-1/2,y) + \dots & \text{on the right.} \end{array} \right.$$

- We define the reflection and transmission matrices

$$R^{\varepsilon} = \begin{pmatrix} r_{11}^{\varepsilon} & r_{12}^{\varepsilon} \\ r_{21}^{\varepsilon} & r_{22}^{\varepsilon} \end{pmatrix} \qquad T^{\varepsilon} = \begin{pmatrix} t_{11}^{\varepsilon} & t_{12}^{\varepsilon} \\ t_{21}^{\varepsilon} & t_{22}^{\varepsilon} \end{pmatrix}.$$

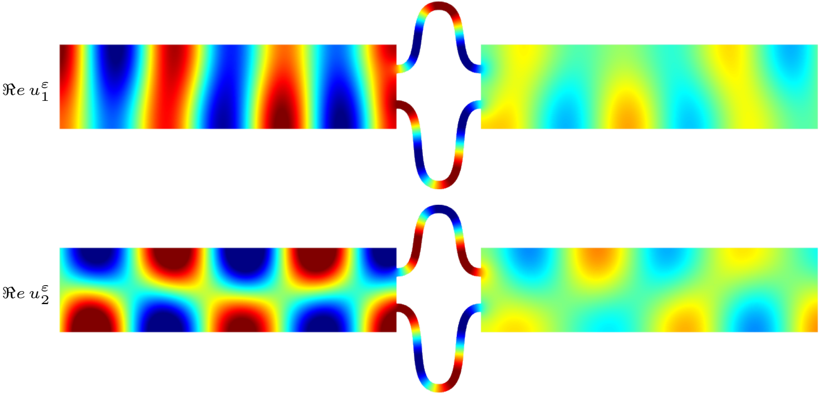
Goal: find a geometry such that:

1) energy is completely transmitted
2) mode 1 is converted into mode 2

$$R^{\varepsilon} \approx \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad T^{\varepsilon} \approx \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Mode converter - geometry

- We decide to work in the following geometry with thin ligaments:



- This may seem **paradoxical** because in general in this Ω, energy is mostly backscattered:

$$R^\varepsilon \approx \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad T^\varepsilon \approx \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \ldots$$

Mode converter - exploiting symmetry

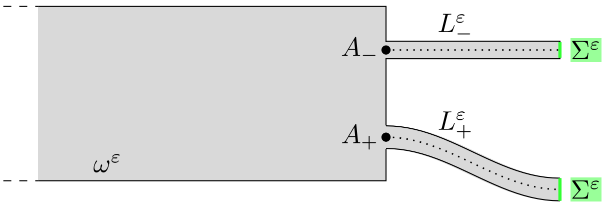
- We impose Ω to be symmetric wrt the (Oy) axis. Then we can show that

$$R^\varepsilon = \frac{R_N^\varepsilon + R_D^\varepsilon}{2} \qquad T^\varepsilon = \frac{R_N^\varepsilon - R_D^\varepsilon}{2}$$

where R_N^ε, R_D^ε are the reflection matrices of the problems

$$(\mathscr{P}_N^\varepsilon)\left|\begin{array}{rl} \Delta u_N^\varepsilon + k^2 u_N^\varepsilon = & 0 \text{ in } \omega^\varepsilon \\ \partial_\nu u_N^\varepsilon = & 0 \text{ on } \partial\omega^\varepsilon \setminus \Sigma^\varepsilon \\ \partial_\nu u_N^\varepsilon = & 0 \text{ on } \Sigma^\varepsilon \end{array}\right. \quad (\mathscr{P}_D^\varepsilon)\left|\begin{array}{rl} \Delta u_D^\varepsilon + k^2 u_D^\varepsilon = & 0 \text{ in } \omega^\varepsilon \\ \partial_\nu u_D^\varepsilon = & 0 \text{ on } \partial\omega^\varepsilon \setminus \Sigma^\varepsilon \\ u_D^\varepsilon = & 0 \text{ on } \Sigma^\varepsilon \end{array}\right.$$

set in the half-waveguide ω^ε (here $\Sigma^\varepsilon := \partial\omega^\varepsilon \setminus \partial\Omega^\varepsilon$):



Mode converter - asymptotic analysis

- In the asympt. analysis of $(\mathscr{P}_N^\varepsilon)$, $(\mathscr{P}_D^\varepsilon)$, we meet the 1D problems:

$$(\mathcal{P}_N^{\pm})\left|\begin{array}{ll}\partial_s^2 v + k^2 v = 0 & \text{in } (0;\ell_\pm)\\ v(0) = \partial_s v(\ell_\pm) = 0 & \end{array}\right. \qquad (\mathcal{P}_D^{\pm})\left|\begin{array}{ll}\partial_s^2 v + k^2 v = 0 & \text{in } (0;\ell_\pm)\\ v(0) = v(\ell_\pm) = 0 & \end{array}\right.$$

We choose $\boldsymbol{k\ell_+ = m\pi}$ and $\boldsymbol{k\ell_- = (m+1/2)\pi}$. In this situation:
- L_+^ε is resonant for the Neumann pb. but not for the Dirichlet one;
- L_-^ε is resonant for the Dirichlet pb. but not for the Neumann one.

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- L_+^ε is resonant for the Neumann pb. but not for the Dirichlet one;
- L_-^ε is resonant for the Dirichlet pb. but not for the Neumann one.

For the Neumann pb., L_+^ε acts at order ε^0 while L_-^ε acts at higher order.
For the Dirichlet pb., L_-^ε acts at order ε^0 while L_+^ε acts at higher order.

$\Rightarrow$ The action of the two ligaments decouple at order ε^0.

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- In the asympt. analysis of $(\mathscr{P}_N^\varepsilon)$, $(\mathscr{P}_D^\varepsilon)$, we meet the 1D problems:

$$(\mathcal{P}_N^\pm)\left|\begin{array}{ll}\partial_s^2 v + k^2 v = 0 & \text{in } (0;\ell_\pm)\\ v(0) = \partial_s v(\ell_\pm) = 0 & \end{array}\right. \qquad (\mathcal{P}_D^\pm)\left|\begin{array}{ll}\partial_s^2 v + k^2 v = 0 & \text{in } (0;\ell_\pm)\\ v(0) = v(\ell_\pm) = 0 & \end{array}\right.$$

We choose $\boldsymbol{k\ell_+ = m\pi}$ and $\boldsymbol{k\ell_- = (m+1/2)\pi}$. In this situation:
- L_+^ε is resonant for the Neumann pb. but not for the Dirichlet one;
- L_-^ε is resonant for the Dirichlet pb. but not for the Neumann one.

For the Neumann pb., L_+^ε acts at order ε^0 while L_-^ε acts at higher order.
For the Dirichlet pb., L_-^ε acts at order ε^0 while L_+^ε acts at higher order.

$\Rightarrow$ The action of the two ligaments decouple at order ε^0.

- With the explicit representation provided by the asymptotic analysis (as for cloaking), we can find **positions** and **lengths** of the ligaments such that

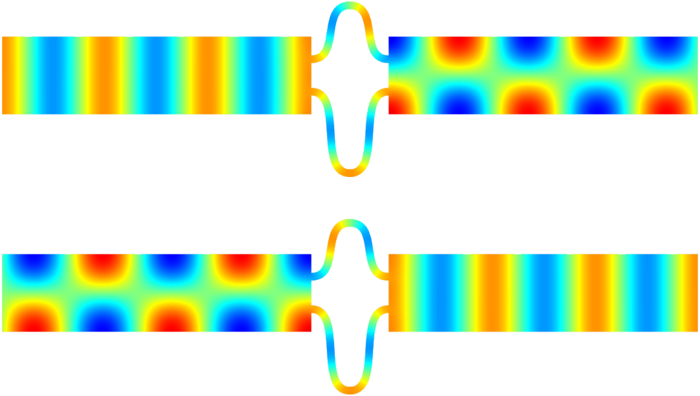
$$R_N^\varepsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + o(1) \qquad R_D^\varepsilon = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} + o(1) \qquad \text{when } \varepsilon \to 0.$$

Mode converter - results

► Thus tuning precisely the positions and lengths of the ligaments, we can ensure absence of reflection and mode conversion:

$t \mapsto \Re e\left(u_1^\varepsilon e^{-i\omega t}\right)$

$t \mapsto \Re e\left(u_2^\varepsilon e^{-i\omega t}\right)$



Conclusion

What we did

♠ We explained how to cloak any object in monomode regime and to design mode converters using thin resonators. Two main ingredients:

- Around resonant lengths, effects of order ε^0 with perturb. of width ε.
- The 1D limit problems in the resonator provide a rather explicit dependence wrt to the geometry.

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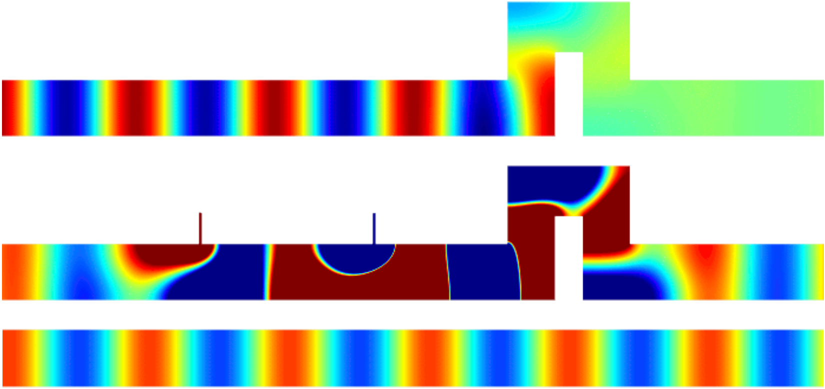
What we did

♠ We explained how to cloak any object in monomode regime and to design mode converters using thin resonators. Two main ingredients:

- Around resonant lengths, effects of order ε^0 with perturb. of width ε.
- The 1D limit problems in the resonator provide a rather explicit dependence wrt to the geometry.

Possible extensions and open questions

1) We can similarly hide penetrable obstacles or work in 3D.
2) We can do cloaking at a finite number of wavenumbers (thin structures are resonant at one wavenumber otherwise act at order ε).
3) With Dirichlet BCs, other ideas must be found.
4) Can we realize exact cloaking ($T = 1$ exactly)? This question is also related to robustness of the device.



Thank you for your attention!

L. Chesnel, J. Heleine and S.A. Nazarov. Design of a mode converter using thin resonant slits. Comm. Math. Sci., vol. 20, 2:425-445, 2022.

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