Logarithmic Schrödinger equation with quadratic potential

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Based on joint works with

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Linear Schrödinger equation: dispersion

$$i\partial_t u + \frac{1}{2}\Delta u = 0, \quad x \in \mathbb{R}^d, \quad u_{|t=0} = u_0 \in \mathcal{S}(\mathbb{R}^d).$$

Explicit solution : $u(t,x) = e^{i\frac{t}{2}\Delta}u_0(x) = \frac{1}{(2i\pi t)^{d/2}}\int_{\mathbb{R}^d} e^{i\frac{|x-y|^2}{2t}}u_0(y)dy.$

Two consequences:

- Dispersion: $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \lesssim \frac{1}{|t|^{d/2}}\|u_0\|_{L^1(\mathbb{R}^d)}.$
- Large time description: $\|u(t) - A(t)u_0\|_{L^2(\mathbb{R}^d)} \underset{t\to\pm\infty}{\longrightarrow} 0$, where

$$A(t)u_0(x) = \frac{1}{(it)^{d/2}}\hat{u}_0\left(\frac{x}{t}\right)e^{i\frac{|x|^2}{2t}}.$$

Nonlinear Schrödinger equation: power nonlinearity

For $\lambda \in \mathbb{R}$, $2/d \leqslant \sigma < \frac{2}{(d-2)_+}$, consider:

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^{2\sigma} u, \quad x \in \mathbb{R}^d, \quad u_{|t=0} = u_0 \in H^1(\mathbb{R}^d).$$

- For small data, global existence ($u \in L^\infty(\mathbb{R}; H^1(\mathbb{R}^d))$), and asymptotically linear behavior,

$$\exists u_+ \in H^1(\mathbb{R}^d), \quad \|u(t) - e^{i\frac{t}{2}\Delta} u_+\|_{H^1(\mathbb{R}^d)} \underset{t\to\infty}{\longrightarrow} 0.$$

- If $\lambda > 0$: same conclusion, no size restriction.
- If $\lambda < 0$: finite time blow-up is possible,

$$\lim_{t \to T^*} \|\nabla u(t)\|_{L^2} = \infty.$$

Solitary waves ($u(t,x) = e^{i\nu t}\phi(x)$) are unstable. They are (orbitally) stable when $0 < \sigma < 2/d$.

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Logarithmic nonlinear Schrödinger equation

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u\,, \quad u_{|t=0} = u_0.$$

⇝ Physical motivation: quantum optics, quantum gravity, BEC, ...

⇝ Formal conservations:

- Mass: $M(u(t)) := \|u(t)\|^2_{L^2(\mathbb{R}^d)}$.
- Energy (Hamiltonian):

$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + \lambda \int_{\mathbb{R}^d} |u(t,x)|^2 \left(\ln|u(t,x)|^2 - 1\right) dx.$$

Many unusual features, due to the singularity of the logarithm at 0.

Example

For $\lambda < 0$, no solution is dispersive (Th. Cazenave '83),

$$\inf_{t\in\mathbb{R}} \inf_{1\leqslant p\leqslant\infty} \|u(t)\|_{L^p(\mathbb{R}^d)} > 0.$$

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$$i\partial_t u + \frac{1}{2}\Delta u = \lambda \ln\left(|u|^2\right) u.$$

- For $k > 0$, ku solves

$$i\partial_t(ku) + \frac{1}{2}\Delta(ku) = \lambda \ln\left(|u|^2\right) ku = \lambda \ln\left(|ku|^2\right) ku - \lambda\left(\ln k^2\right) ku.$$

Gauge transform: $u_k := (ku)e^{2it\lambda \ln k}$ solves the same equation as $u(= u_1)$.

$\rightsquigarrow$ A scaling factor does not change the dynamics.

- If $u_{|t=0}$ is Gaussian, then $u(t,\cdot)$ is Gaussian for all $t \in \mathbb{R}$: ODEs in time.
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Gaussons

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Let $\nu \in \mathbb{R}$: $\phi_\nu(x) = e^{-\frac{\nu}{2\lambda}+d/2} e^{\lambda |x|^2}$. Standing wave $u(t,x) = \phi_\nu(x) e^{i\nu t}$.

Theorem (Cazenave '83, Ardila '16)

Let $d \geqslant 1$ and $\lambda < 0$. The Gausson is orbitally stable in the energy space W: For any $\varepsilon > 0$, there exists $\eta > 0$ such that if $u_0 \in W$ satisfies $\|u_0 - \phi_\nu\|_W < \eta$, then the solution u exists for all $t \in \mathbb{R}$, and

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \inf_{y\in\mathbb{R}^d} \|u(t) - e^{i\theta}\phi_\nu(\cdot - y)\|_W < \varepsilon.$$

Crucial ingredient: variational characterization based on logarithmic Sobolev inequality.

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Crucial ingredient: variational characterization based on logarithmic Sobolev inequality.

Variational analysis

$$-\frac{1}{2}\Delta\phi + \nu\phi + \lambda\phi \ln|\phi|^2 = 0.$$

Action:

$$S_\nu(u) = E(u) + \nu\|u\|_{L^2}^2 = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \lambda \int |u|^2 \left(\ln|u|^2 - 1\right) + \nu\|u\|_{L^2}^2.$$

Nehari functional: $I_\nu(u) = 2S_\nu(u) + 2\lambda\|u\|_{L^2}^2$.

Set of ground states:

$$\mathcal{G}_\nu = \{\phi \in W \setminus \{0\} \mid I_\nu(\phi) = 0, S_\nu(\phi) = D(\nu)\},$$

$$\text{where } D(\nu) = \inf\{S_\nu(u) \mid u \in W \setminus \{0\}, I_\nu(u) = 0\}$$

$$= \inf\{-\lambda\|u\|_{L^2}^2 \mid u \in W \setminus \{0\}, I_\nu(u) = 0\}.$$

Proposition (Ardila '16)

$$D(\nu) = -\lambda\pi^{d/2}(-2\lambda)^{-d/2}e^{-\nu/\lambda+d}; \quad \mathcal{G}_\nu = \{e^{i\theta}\phi_\nu(\cdot - y);\ \theta \in \mathbb{R},\ y \in \mathbb{R}^d\}.$$

Logarithmic Sobolev inequality

Theorem

Let $f \in H^1(\mathbb{R}^d)$, $a > 0$.

$$\int_{\mathbb{R}^d} |f(x)|^2 \ln \left(\frac{|f(x)|^2}{\|f\|_{L^2}^2} \right) dx \leqslant \frac{a^2}{\pi} \int_{\mathbb{R}^d} |\nabla f|^2 - d(1 + \ln a)\|f\|_{L^2}^2.$$

Moreover, there is equality iff f *is, up to translation, a multiple of* $e^{-\pi|x|^2/(2a^2)}$.

Remark

The left hand side is $\int_{\mathbb{R}^d} |f|^2 \ln |f|^2$, not $\int_{\mathbb{R}^d} |f|^2 \left|\ln |f|^2\right|$.

Introducing a quadratic potential

$$i\partial_t u + \frac{1}{2}\Delta u = V(x)u + \lambda \ln\left(|u|^2\right) u.$$

Theorem (Ardila-Cely-Squassina '20)

Let $d \geqslant 1$ and $\lambda < 0$. Suppose that

$$V(x) = \frac{\kappa(\kappa+2\lambda)}{2}|x|^2, \quad \kappa > -2\lambda > 0 \text{ (confining potential).}$$

The (generalized) Gausson is $\phi_\nu(x) = e^{-\frac{\nu+\kappa d/2}{2\lambda}} e^{-\kappa|x|^2/2}$, $\nu \in \mathbb{R}$. Associated standing wave $u(t,x) = \phi_\nu(x)e^{i\nu t}$, which is orbitally stable in the energy space $\Sigma = H^1 \cap \mathcal{F}(H^1)$: For any $\varepsilon > 0$, there exists $\eta > 0$ such that if $u_0 \in \Sigma$ satisfies $\|u_0 - \phi_\nu\|_\Sigma < \eta$, then

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\phi_\nu\|_\Sigma < \varepsilon.$$

Remark

Space translation invariance is lost.

Remark

The scheme and ingredients of the proof are somehow similar.

Questions considered:

- What if $\lambda > 0$?
- What if V is anisotropic?
- What if V is quadratic repulsive?

Singular properties

$$i\partial_t u + \frac{1}{2}\Delta u = V(x)u + \lambda \ln\left(|u|^2\right) u.$$

• The size invariance remains: if u is a solution, then so is $(ku)e^{it\lambda \ln |k|^2}$, for any $k \in \mathbb{C}$.

• Tensorization: if $V(x) = \sum_{j=1}^{d} V_j(x_j)$, and $u_0(x) = \prod_{j=1}^{d} u_{0j}(x_j)$, then

$u(t,x) = \prod_{j=1}^{d} u_j(t,x_j)$, where each u_j solves a one-dimensional equation.

• When V is quadratic, an initial Gaussian propagates as a Gaussian. Link with a linear phenomenon: for

$$i\partial_t u + \frac{1}{2}\Delta u = V(t,x)u, \quad V(t,x) = \langle x, M(t)x\rangle, \ M \in \mathbb{R}^{d\times d},$$

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$\ln e^{-\langle x, \mathrm{Re}\, A(t)x\rangle} = -\langle x, \mathrm{Re}\, A(t)x\rangle \dots$

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Case $\lambda > 0$ (with Guillaume Ferriere)

Theorem (RC-G. Ferriere '21)

Let $d \geqslant 1$ and $\lambda > 0$. Suppose that

$$V(x) = \frac{\kappa(\kappa + 2\lambda)}{2}|x|^2, \quad \kappa > 0 \text{ (confining potential).}$$

The (generalized) Gausson is $\phi_\nu(x) = e^{-\frac{\nu+\kappa d/2}{2\lambda}} e^{-\kappa|x|^2/2}$, $\nu \in \mathbb{R}$. Standing wave $u(t,x) = \phi_\nu(x)e^{i\nu t}$, which is orbitally stable in the energy space: For any $\varepsilon > 0$, there exists $\eta > 0$ such that if $u_0 \in \Sigma$ satisfies $\|u_0 - \phi_\nu\|_\Sigma < \eta$, then

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\phi_\nu\|_\Sigma < \varepsilon.$$

Main ingredients of the proof

Variational analysis, somehow like Ardila. The logarithmic Sobolev inequality is replaced by:

Lemma

Let $f \in \mathcal{F}(H^1(\mathbb{R}^d))$ and $a > 0$:

$$-\int_{\mathbb{R}^d} |f(x)|^2 \ln \left(\frac{|f(x)|^2}{\|f\|_{L^2}^2} \right) dx \leqslant a \int_{\mathbb{R}^d} |x|^2 |f(x)|^2 dx + \frac{d}{2} \|f\|_{L^2}^2 \ln \frac{\pi}{a}.$$

There is equality if and only if $|f(x)| = ce^{-a|x|^2/2}$, with $c = \|f\|_{L^2} (a/\pi)^{d/2}$.

Remark (Anisotropic harmonic potential)

Theorem and Lemma readily adapted to the case

$V(x) = \sum_{j=1}^{d} \frac{\kappa_j(\kappa_j + 2\lambda)}{2} x_j^2$, $\kappa_j > 0$. False if at least one $\kappa_j = 0$ (wait and see).

Proof of the lemma

For $a > 0$, consider the normalized Gaussian

$$\nu_a(x) = \left(\frac{a}{\pi}\right)^{d/2} e^{-a|x|^2} : \quad \int_{\mathbb{R}^d} \nu_a(x)dx = 1.$$

Csiszár-Kullback inequality: μ, ν probability densities,

$$\|\mu - \nu\|^2_{L^1(\mathbb{R}^d)} \leqslant 2 \int_{\mathbb{R}^d} \mu(x) \ln\left(\frac{\mu(x)}{\nu(x)}\right) dx.$$

Consider $\mu(x) = |f(x)|^2/\|f\|^2_{L^2}$ and $\nu = \nu_a$: use the positivity of the relative entropy, and expand.
Equality iff $\mu = \nu_a$.

Partial confinement: $d \geqslant 2$

$$i\partial_t u + \frac{1}{2}\Delta u = V(x)u + \lambda \ln\left(|u|^2\right) u.$$

We assume $\lambda > 0$.
$(x', x'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, with $d_1, d_2 \geqslant 1$, $d_1 + d_2 = d$, and

$$V(x', x'') = \frac{\omega^2}{2}|x'|^2.$$

Tensorization: in the case of tensorized data, the solution in x'' is dispersive, hence no standing wave.

Theorem (RC-G. Ferriere '21)

Let $(x', x'') \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, *with* $d_1, d_2 \geqslant 1$, $d_1 + d_2 = d$, *and*

$$V(x', x'') = \frac{\omega^2}{2}|x'|^2, \quad \omega > 0.$$

Suppose $\lambda > 0$. *Let* $u_0 \in \Sigma \setminus \{0\}$, *and* $\gamma(x'') := e^{-|x''|^2/2}$. *Introduce*

$$\rho(t, y) := \tau(t)^{d_2} \int_{\mathbb{R}^{d_1}} \left|u\left(t, x', y\tau(t)\right)\right|^2 dx' \times \frac{\pi^{d_2}}{\|u_0\|^2_{L^2(\mathbb{R}^d)}},$$

where τ *solves a precise ODE, and satisfies* $\tau(t) \sim 2t\sqrt{\lambda \ln t}$ *as* $t \to \infty$.

$$\int_{\mathbb{R}^{d_2}} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \rho(t, y) dy \underset{t\to\infty}{\longrightarrow} \int_{\mathbb{R}^{d_2}} \begin{pmatrix} 1 \\ y \\ |y|^2 \end{pmatrix} \gamma^2(y) dy,$$

and

$$\rho(t, \cdot) \underset{t\to\infty}{\rightharpoonup} \gamma^2 \quad \text{weakly in } L^1(\mathbb{R}^{d_2}).$$

Solitary waves & repulsive potential (with Chunmei Su)

$$i\partial_t u + \frac{1}{2}\Delta u = -\omega^2\frac{|x|^2}{2}u + \lambda u \ln\left(|u|^2\right), \quad x \in \mathbb{R}^d, \quad \omega > 0.$$

Theorem (RC-C. Su '21)

Let $-\lambda > \omega > 0$. There exist two (generalized) Gaussons,

$$\phi_{k_\pm}(x) = e^{-\frac{dk_\pm}{4\lambda}} e^{-k_\pm |x|^2/2}, \quad \textit{where} \quad k_\pm = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Each stationary solution generates a continuous family of solitary waves,

$$u_{\pm,\nu}(t,x) = \phi_{k_\pm,\nu}(x)e^{i\nu t}, \quad \phi_{k_\pm,\nu}(x) = e^{-\frac{\nu}{2\lambda}}\phi_{k_\pm}(x), \quad \nu \in \mathbb{R}.$$

ϕ_{k_-} and ϕ_{k_+} are two positive solutions to the stationary equation

$$-\frac{1}{2}\Delta\phi - \omega^2\frac{|x|^2}{2}\phi + \lambda\phi\ln\left(|\phi|^2\right) = 0.$$

Stability

Linearizing around ϕ_k, for $k = k_-$ or k_+, leads to:

$$i\partial_t u + \frac{1}{2}\Delta u = -\omega^2 \frac{|x|^2}{2} u - \frac{dk}{2} u - \lambda k |x|^2 u = k^2 \frac{|x|^2}{2} u - \frac{dk}{2} u.$$

Shifted harmonic oscillator, $H_k = -\frac{1}{2}\Delta + k^2 \frac{|x|^2}{2} - \frac{dk}{2}$, whose point spectrum is $k\mathbb{N}$: linear and spectral stability.

Definition

A standing wave $u(t,x) = \phi(x)e^{i\nu t}$ solution is orbitally stable in the energy space if for any $\varepsilon > 0$, there exists $\eta > 0$ such that if $u_0 \in \Sigma$ satisfies $\|u_0 - \phi\|_\Sigma < \eta$, then the solution u exists for all $t \in \mathbb{R}$, and

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\phi\|_\Sigma < \varepsilon.$$

Otherwise, the standing wave is said to be unstable.

Nonlinear instability

Theorem (RC-C. Su '21)

Let $-\lambda > \omega > 0$.

$$u_{\pm,\nu}(t,x) = \phi_{k_\pm,\nu}(x)e^{i\nu t}, \quad \phi_{k_\pm,\nu}(x) = e^{-\frac{\nu}{2\lambda}}\phi_{k_\pm}(x), \quad \nu \in \mathbb{R}.$$

Every such solitary wave is unstable.

$\rightsquigarrow$ Unlike in the case of a power-nonlinearity, not by blow-up: all solutions are global in time.

$\rightsquigarrow$ Instability by small initial translations (in space or frequency).

$\rightsquigarrow$ $u_{-,\nu}$ is unstable even if we restrict the analysis to radial solutions: consider centered Gaussians, and linearize the ODEs.

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On the notion of ground state

Most standard notions of ground state:

- Minimizer of the action $E + \nu M$.
- Minimizer of the energy E for a given mass M.
- Positive solution of $dE + \nu dM = 0$.

$\rightsquigarrow$ In the case of homogeneous nonlinearities, the three notions coincide.

$\rightsquigarrow$ None of these notions is satisfied in the present case:

- ϕ_{k_-} and ϕ_{k_+} are such that $dE = 0$.
- E unbounded from below for any given mass: space translations.
- Introducing the action and the Nehari functional (like before), $D(\nu) = 0$ and $\mathcal{G}(\nu) = \emptyset$.

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