Méthodes numériques pour des problèmes inverses en mécanique des fluides

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A unique continuation problem

We look for $(u, p) \in H^1(\Omega)^d \times L^2_0(\Omega)$ such that

$$\left\{ \begin{array}{rcll} -\nu\Delta u + \nabla p & = & 0 & \text{in } \Omega \\ \nabla \cdot u & = & 0 & \text{in } \Omega \end{array} \right.$$

and

$$u = u_{\mathrm{M}} \quad \text{in} \quad \omega_{\mathrm{M}}$$

where $\omega_{\mathrm{M}} \subset \Omega$ is an open domain.

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Unique continuation property:

$$u_{\mathrm{M}} = 0 \text{ in } \omega_{\mathrm{M}} \Rightarrow (u, p) = (0, 0) \text{ in } H^1(\Omega)^d \times L^2_0(\Omega).$$

[Fabre, Lebeau (1996)]

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Stability?

Ill-posedness of inverse problems

- The solution of inverse problems generally does not depend continuously on the measurements
- Inverse problems are **ill-posed**.
- Continuity is restored in the presence of an a priori bound on the solution.
- We talk about **conditional stability** properties.

Conditional stability for the continuation problem

Let $K \subset\subset \Omega$. There exists $C > 0$ and $\tau \in (0, 1)$ such that, for all $(u, p) \in H^1(\Omega)^d \times H^1(\Omega)$ solution of the homogeneous Stokes equation such that $\|u\|_{L^2(\Omega)} \leq M$

$$\|u\|_{L^2(K)} \leq CM^{1-\tau} \|u\|^{\tau}_{L^2(\omega_{\mathrm{M}})}.$$

[Lin, Uhlmann, Wang (2010)]

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$$\|u\|_{L^2(K)} \leq CM^{1-\tau} \|u\|^{\tau}_{L^2(\omega_{\mathrm{M}})}.$$

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Remarks

These inequalities are proven thanks to three-balls inequalities

$$\|u\|_{L^2(B(R_2))} \leq C \|u\|^{\tau}_{L^2(B(R_1))} \|u\|^{1-\tau}_{L^2(B(R_3))} \text{ for } R_1 < R_2 < R_3.$$

[Alessandrini, Rondi, Rosset, Vessella (2009)]

A classical strategy for the numerical resolution: Tikhonov regularization

We consider the functional:

$$J_\alpha(v) = \frac{1}{2}\|u(v) - u_{\mathrm{M}}\|^2_{L^2(\omega_{\mathrm{M}})}$$

where $u(v)$ satisfies

$$\left\{ \begin{array}{rcll} -\nu\Delta u + \nabla p & = & f & \text{in } \Omega \\ \nabla \cdot u & = & 0 & \text{in } \Omega \\ u & = & v & \text{on } \partial\Omega \end{array} \right.$$

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Choice of α ? choice of $\|\cdot\|_{\partial\Omega}$?

- Add of an a priori.
- Adaptative choice of α with respect to the noise or to the mesh size.
  - Morozov criteria: for u^δ_{M} a noisy data, we choose α such that

    $$\|u(v^\delta_\alpha) - u^\delta_{\mathrm{M}}\|_{L^2(\omega_{\mathrm{M}})} \simeq \delta$$

    where v^δ_α minimizes J_α.
  - Balance between the discretization error and the regularization error.
    [Burman, Hansbo, Larson (2016)]

The discretize-then-regularize strategy

In what follows, we consider another strategy consisting of discretizing first then regularizing the discrete problem by adding stabilisation terms.
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Stabilisation methods

- designed to sort out stability issues for discretized problems
- formed by adding terms to the discrete Galerkin formulation
- vanish quickly enough so that optimal error estimates can be obtained
- originally developed for advection-diffusion equations, fluid equations

  [Brooks, Hughes (1981)], [Hughes, Franca, Balestra (1986)], [Johnson, Nävert, Pitkäranta (2016)]

Variational formulation

[M.B., Burman, Fernandez, Voisembert (2021)]

We define:

$$V := [H^1(\Omega)]^d, \quad V_0 := [H^1_0(\Omega)]^d, \quad L_0 := L^2_0(\Omega), \quad \text{and} \quad L := L^2(\Omega)$$

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$$V := [H^1(\Omega)]^d, \quad V_0 := [H_0^1(\Omega)]^d, \quad L_0 := L_0^2(\Omega), \quad \text{and} \quad L := L^2(\Omega)$$

We look for $(u, p) \in V \times L_0$ such that

$$a(u, v) - b(p, v) + b(q, u) = (f, v)_{L^2(\Omega)}, \quad \forall (v, q) \in V_0 \times L$$

where

$$a(u, v) := \nu \int_\Omega \nabla u : \nabla v \quad \text{and} \quad b(p, v) := \int_\Omega p \nabla \cdot v.$$

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We define $A : (V \times L_0) \times (V_0 \times L)$ by

$$A[(u, p), (v, q)] := a(u, v) - b(p, v) + b(q, u)$$

A mixed formulation of the minimization problem

We want to find $(u, p) \in V \times L_0$ that minimizes the functional

$$J(u) = \frac{1}{2} \int_{\omega_{\mathrm{M}}} |u - u_{\mathrm{M}}|^2$$

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We introduce the Lagrangian in $(V \times L_0) \times (V_0 \times L)$

$$\mathcal{L}[(u, p), (z, y)] = J(u) + A[(u, p), (z, y)] - (f, z)_{L^2(\Omega)}$$

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$$\begin{cases} A[(u,p),(w,x)] = & (f, w)_{L^2(\Omega)} \\ A[(v,q),(z,y)] + (u,v)_{L^2(\omega_{\mathrm{M}})} = & (u_{\mathrm{M}}, v)_{L^2(\omega_{\mathrm{M}})} \end{cases}$$

for all $[(v,q),(w,x)] \in (V \times L_0) \times (V_0 \times L)$

[Brezzi, Fortin (1991)], [Bourgeois, Recoquillay (2018)]

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This problem is ill-posed!

Minimization problem at the discrete level

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We modify the Lagrangian: for all $(u_h, p_h) \in V_h \times Q_h^0$ and $(z_h, y_h) \in W_h \times Q_h$

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for all $(v_h, q_h) \in V_h \times Q_h^0$ and $(w_h, x_h) \in W_h \times Q_h$

Properties of the stabilising terms

Choice of the stabilisation terms S and S^*

- They allow to get a well-posed problem at the discrete level.
- They are consistent with the continuous formulation in the sense that, for the solution of the problem $[(u,p),(z,y)]$, we have

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$$S^*[(z,y),(w,x)] = 0 \text{ and } S[(u,p),(v,q)] = 0$$

At the discrete level, if the measurements are exact in the sense that $u_{\mathrm{M}} = u|_{\omega_{\mathrm{M}}}$, we have

$$(z,y) = (0,0).$$

We will assume that (u,p) belongs to $[H^2(\Omega)]^d \times H^1(\Omega)$.

Design of the stabilising terms

$$S[(u_h, p_h), (v_h, q_h)] = \gamma_u \sum_{F \in \mathcal{F}_i} \int_F h_F [\![\nabla u_h]\!][\![\nabla v_h]\!] + \gamma_{div} \int_\Omega (\nabla \cdot u_h)(\nabla \cdot v_h)$$

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and

$$S^*[(z_h,y_h),(w_h,x_h)] = \gamma_u^* \int_\Omega \nabla z_h : \nabla w_h + \gamma_p^* \int_\Omega y_h x_h$$

Well-posedness of the discrete problem

We set

$$\mathcal{A}\big([(u_h,p_h),(z_h,y_h)],[(v_h,q_h),(w_h,x_h)]\big)$$

$$= A[(u_h,p_h),(w_h,x_h)] - S^*[(z_h,y_h),(w_h,x_h)] + A[(v_h,q_h),(z_h,y_h)] + (u_h,v_h)_{L^2(\omega_{\mathrm{M}})}$$

$$+S[(u_h,p_h),(v_h,q_h)]$$

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We take $[(v_h,q_h),(w_h,x_h)] = [(u_h,p_h),(-z_h,-y_h)]$

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Well-posedness of the discrete problem

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$$\mathcal{A}([(u_h,p_h),(z_h,y_h)],[(v_h,q_h),(w_h,x_h)])$$

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We have

$$S^*[(z_h,y_h),(z_h,y_h)] \geq C(\|z_h\|^2_{V_0} + \|y_h\|^2_L)$$

and

$$(u_h,u_h)_{L^2(\omega_{\mathrm{M}})} + S[(u_h,p_h),(u_h,p_h)] \geq Ch^2(\|u_h\|^2_V + \|p_h\|^2_L)$$

Error estimate

Error estimate

Let $f \in L^2(\Omega)$ be given. We assume that $u_{\mathrm{M}} := u|_{\omega_{\mathrm{M}}}$. We assume that the solution (u, p) belongs to $[H^2(\Omega)]^d \times H^1(\Omega)$. Then, for all $K \subset\subset \Omega$, there exists $\tau \in (0, 1)$ such that

$$\|u - u_h\|_{L^2(K)} \leq Ch^\tau \left(\|u\|_{[H^2(\Omega)]^d} + \|p\|_{H^1(\Omega)}\right) + h\|f\|_{L^2(\Omega)}.$$

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Error estimate in presence of noise

Let $f \in L^2(\Omega)$ and $\delta u \in L^2(\omega_{\mathrm{M}})$ be given. We assume that $u_{\mathrm{M}} := u|_{\omega_{\mathrm{M}}} + \delta u$. We assume that the solution (u, p) belongs to $[H^2(\Omega)]^d \times H^1(\Omega)$. Then, for all $K \subset\subset \Omega$, there exists $\tau \in (0, 1)$ such that

$$\|u - u_h\|_{L^2(K)} \leq Ch^{\tau}(\|u\|_{[H^2(\Omega)]^d} + \|p\|_{H^1(\Omega)} + h^{-1}\|\delta u\|_{L^2(\omega_{\mathrm{M}})}) + h\|f\|_{L^2(\Omega)}.$$

Error estimate

Remarks

- No error estimate for the pressure.
- No global estimates.

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Theoretical study

- **Error estimate for the pressure:** for $(u,p) \in H^1(\Omega) \times L^2(\Omega)$ such that $\|u\|_{H^1(\Omega)} + \|p\|_{L^2(\Omega)} \leq M$

$$\|u\|_{H^1(K)} + \|p\|_{L^2(K)} \leq CM^{1-\tau} \left(\|u\|_{H^1(\omega_M)} + \|p\|_{L^2(\omega_M)} \right)^{\tau}.$$

  [M.B., Egloffe, Grandmont (2013)], [Badra, Caubet, Dardé (2016)]

- **Global estimates:** for $(u,p) \in H^2(\Omega) \times H^1(\Omega)$ such that $\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)} \leq M$

$$\|u\|_{L^2(\Omega)} \leq C \frac{M}{\log\left(1 + \frac{M}{\|u\|_{L^2(\omega_M)}}\right)}$$

  [Badra, Caubet, Dardé (2016)]

Error curves

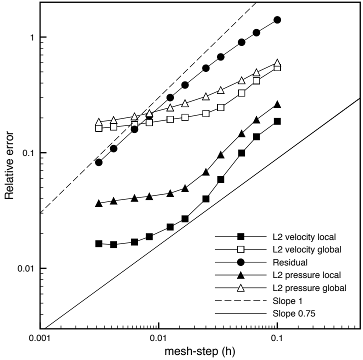


Figure: Error without noise

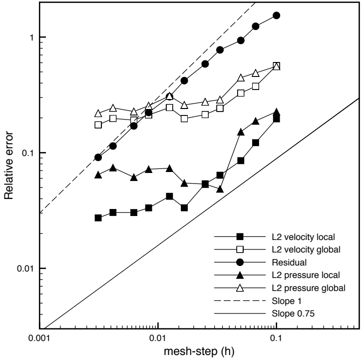
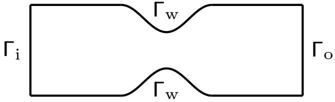


Figure: Error with 10% noise

A non-stationary problem

- Blood flow in a stenotic blood vessel



- Measurements of the velocity on the whole domain (corresponding to 4D-MRI).
  We denote by u^n_{M} the measurement of the velocity at time $t_n = n\Delta t$ in Ω.

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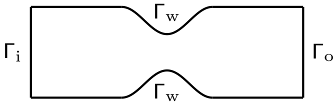

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$$\delta p = \frac{1}{|\Gamma_{\mathrm{i}}|} \int_{\Gamma_{\mathrm{i}}} p - \frac{1}{|\Gamma_{\mathrm{o}}|} \int_{\Gamma_{\mathrm{o}}} p.$$

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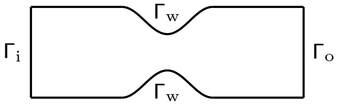

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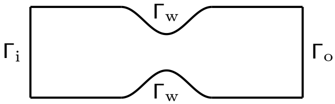

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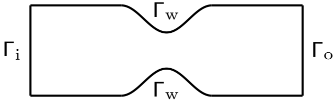

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  [Bertoglio, Nunez, Galarce, Nordsletten, Osses (2018)]
- Resolution of the inverse problem directly on the nonstationary problem
  [Bellassoued, Imanuvilov, Yamamoto (2016), M.B. (2016)]

A direct method combined with a data assimilation method

At each time t_n, we look for (u^n, p^n) which minimizes

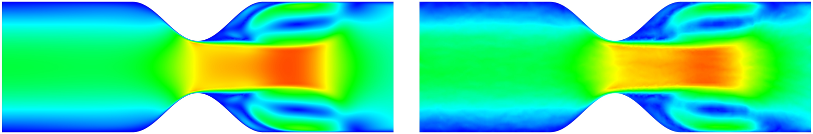
$$\int_\Omega |u^n - u_{\mathrm{M}}^n|^2 \, dx$$

under the constraint that it satisfies the Oseen equations

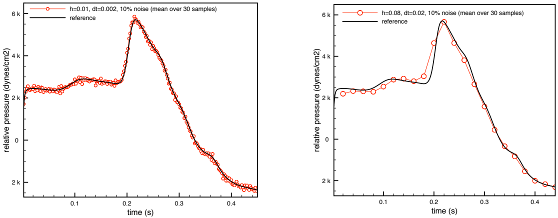
$$\begin{cases} (u_{\mathrm{M}}^n \cdot \nabla) u^n - \nu \Delta u^n + \nabla p^n = -\dfrac{u_{\mathrm{M}}^{n+1} - u_{\mathrm{M}}^n}{\Delta t} & \text{in} \quad \Omega, \\ \nabla \cdot u^n = 0 & \text{in} \quad \Omega. \end{cases}$$

Numerical results

[M.B., Burman, Fernandez, Voisembert (2021)]



Velocity magnitude.
Left: reference, right: reconstruction with space-time subsampling and 10% of noise



RPD (black line: reference, red dotted line: reconstruction)
Left: with 10% of noise, right: with space-time subsampling and 10% of noise

Taking into account the motion of the vessel

Work in progress with M. Abgalessi, M.A. Fernandez, D. Lombardi and M. Nechita

- For realistic data, it is important to take into account the wall motion.
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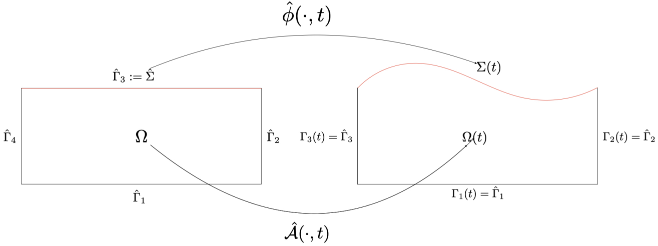

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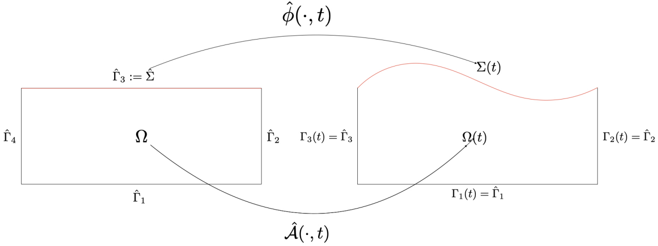

- We do not have boundary conditions on the inlet and outlet.

The fluid-structure interaction model

$$\begin{cases} \rho^{\mathrm{f}} \left(\partial_t \boldsymbol{u}_{\mathrm{f}} + \boldsymbol{u}_{\mathrm{f}} \cdot \nabla \boldsymbol{u}_{\mathrm{f}} \right) - \nabla \cdot \sigma(\boldsymbol{u}_{\mathrm{f}}, p) = 0 & \text{in } \Omega(t), \\ \nabla \cdot \boldsymbol{u}_{\mathrm{f}} = 0 & \text{in } \Omega(t), \\ \sigma(\boldsymbol{u}_{\mathrm{f}}, p) = & \nu\epsilon(\boldsymbol{u}_{\mathrm{f}}) - p\mathrm{Id} \end{cases}$$

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Remarks

Now, the pressure constant is fixed under the condition that the external pressure is known.

Luenberger observer for FSI

In the model, we add filters that involve the measurements.

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[Bertoglio, Chapelle, Fernandez, Gerbeau, Moireau (2013)]

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Boundary conditions in the outlet and inlet?

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[Bertoglio, Chapelle, Fernandez, Gerbeau, Moireau (2013)]

Boundary conditions in the outlet and inlet? We set:

$$\boldsymbol{u}_{\mathrm{f}} = \bar{\boldsymbol{u}}_{\mathrm{f}} \text{ on } \hat{\Gamma}_2 \cup \hat{\Gamma}_4$$

The continuation step

At each time t_n, we look for $(\bar{\boldsymbol{u}}_{\mathrm{f}}{}^n, \bar{p}^n)$ which minimizes

$$\int_{\Omega^{n-1}} |\bar{\boldsymbol{u}}_{\mathrm{f}}{}^n - \boldsymbol{u}^n_{\mathrm{f,M}}|^2 \, dx$$

under the constraint that it satisfies the Oseen equations

$$\begin{cases} \dfrac{1}{\Delta t}\bar{\boldsymbol{u}}_{\mathrm{f}}{}^n + (\boldsymbol{u}^n_{\mathrm{f,M}} \cdot \nabla)\bar{\boldsymbol{u}}_{\mathrm{f}}{}^n - \nu\Delta\bar{\boldsymbol{u}}_{\mathrm{f}}{}^n + \nabla\bar{p}^n = \dfrac{1}{\Delta t}\boldsymbol{u}^{n-1}_{\mathrm{f}} & \text{in} \quad \Omega^{n-1}, \\ \nabla \cdot \bar{\boldsymbol{u}}_{\mathrm{f}}{}^n = 0 & \text{in} \quad \Omega^{n-1}, \end{cases}$$

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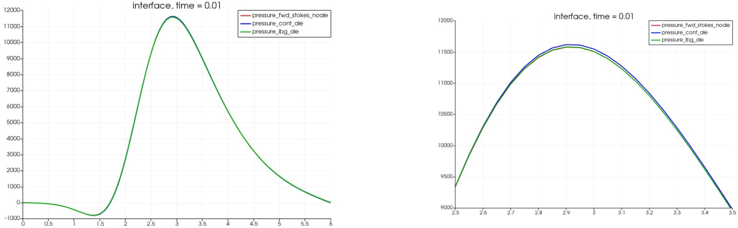
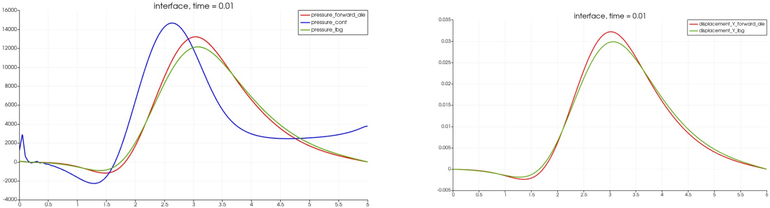
Numerical results: Stokes and Navier-Stokes equation



Stokes equation: recontruction of the pressure on the interface at time 0.1s.



Navier-Stokes equation: recontruction of the pressure and the displacement on the interface at time 0.1s.

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- Comparison of the different ways to combine the steps
- Numerical simulations in 3D
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Thank you for your attention!